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ELASTIC-PLASTIC RESPONSE OF A SLAB TO A HEAT PULSE

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Yip-Chie Lee
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Abstract

The objective of the present study is to establish thermodynamically valid, non-isothermal stress strain relations for the elastic-plastic range, and to obtain solutions of typical one-dimensional problems involving unsteady temperatures.

In the first part of the report the entropy balance equation and the expression for production of internal entropy are considered. The limitations imposed by the second law of thermodynamics upon plastic flow rules are investigated, and subsequently a set of non-isothermal plastic stress strain relations is introduced. It is then shown that these relations are in accord with thermodynamic irreversibility whenever the yield criteria of Tresca and von Mises are used. Moreover, it is found that the stress strain relations of von Mises are a special case of the non-isothermal stress strain relations proposed here. An additional problem concerns the specification of elastic unloading from a plastic state, and in the presence of a temperature dependent yield stress. Precise criteria for elastic loading and repeated plastic

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flow are presented, thus completing the characterization of the elastic-plastic response.

The second part of the report is concerned with an application of the general theory to a problem for the infinite half-space constrained against lateral motion, and subjected to a heat pulse uniformly applied over its boundary. In the analysis of this problem the yield criterion of Tresca is used; the transient and steady state solutions for both strain-hardening and perfectly plastic media, having either a constant or a temperature dependent yield stress, are presented. The appearance of elastic and plastic regions of loading and unloading is studied in some detail, and the residual stresses and deformations are correlated to the maximum boundary temperature.

The solution of the half-space problem is then extended to related problems for an infinite plate of finite thickness, and constrained in the lateral direction. It is shown that expressions for predicting maximum residual stresses and strains can be obtained directly from the solution of the half-space problem.

Numerical results are given for the case when the material is an aluminum alloy. In addition, the maximum boundary temperature corresponding to the recurrence of plastic flow during a second heat pulse is determined assuming that the conditions produced by the first pulse have reached a steady state.

Although the problem selected to illustrate the use and implications of non-isothermal plastic stress strain relations is one-dimensional, the method of approach and the basic equations may be used in the analysis of thermal stress problems in two or three dimensions.

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$\bar{q}$, q_i	heat flux vector and its components
$\bar{n}$, n_i	normal vector and its components
P	mechanical power
K	kinetic energy
ρ	mass density
Λ	material time rate of total internal energy
$\bar{v}$, v_i	velocity vector and its components
$\bar{a}$, a_i	acceleration vector and its components
$\bar{T}$, T_i	surface traction and its components
$\bar{f}$, f_i	body force and its components
τ_{ij}, v_{ij}	total and deviatoric stress tensors
d_{ij}	deformation rate tensor
w_{ij}	vorticity tensor
λ	rate of internal energy density
$\dot{U}$	time rate of elastic internal energy
ϕ	time rate of internal energy due to plastic deformation
$\dot{S}_e$	external entropy flux
s, s_e, s_i	entropy density, entropy production and external entropy density
ε_{ij}, r_{ij}	tensors of elastic and plastic strain
s_{ij}	plastic strain rate
ϑ	volume dilation
F	yield function
ξ	strain hardening parameter

α	coefficient of thermal expansion
t	time
T_o	reference temperature
T	temperature increment over T_o
t_m, t_{mx}	the instants for which the temperatures at the boundary $x = a$ and at the position x are at their maximum
T_m, T_{mL}	maximum boundary temperature for a half-space and for a plate
T_{mx}	maximum temperature attainable at position x
$\tau = \tau_{yy} = \tau_{zz}$	non-vanishing stress components
$\epsilon = \epsilon_{xx}$	non-vanishing strain components
$U = U_x$	non-vanishing displacement component
q	maximum shearing stress
p	maximum shearing strain
ρ_1, ρ_2, ρ_3	transient interface positions
R_1, R_{11}	position of the boundary of the total plastically deformed and steady state plastic regions
t_1, t_{1x}	time instant corresponding to incipient plastic flow at the boundary $x = a$ and at position x
T_1, T_{1x}, T_{mR_1}	temperatures corresponding to incipient plastic flow in a half-space and in a plate
y_c, y_I, y_o	current yield stress, initial yield stress, and yield stress at T_o
E_1	Young's modulus
ν_1	Poisson's ratio

$\mu_E = \frac{E_1}{2(1+\nu_1)}$	elastic modulus of shear
μ_p	plastic modulus of shear
T_M, T_{ML}	maximum boundary temperature for which plastic flow during unloading will impend in a half-space and in a plate
t_{pa}, t_{px}	time instant at which plastic flow during unloading impends at the boundary $x = a$ and at the position x
τ_R, τ_{Rmax}	residual stress in half-space and maximum residual stress in plate
ϵ_R, ϵ_{Rmax}	residual strain in half-space and maximum residual strain in plate
U_R	permanent deformation in half-space
L	thickness of plate
γ	parameter characterizing the shape of the heat pulse in plate problems
T_{mo}	maximum temperature attainable at the insulated boundary of a plate
T_{c1}, T_{c2}	critical maximum boundary temperature for an insulated plate
T_{1x}'	temperature corresponding to incipient plastic flow due to a second heat pulse

PART I. THERMODYNAMICS AND THERMOPLASTICITY

Chapter 1. Introduction and Review

1-1 Introduction

The present theory of plasticity is applicable only to problems in which the effects of temperature on the plastic stress strain relations may be neglected. In particular, the yield function or functions which characterize yielding are assumed to be independent of temperature. The reason for the neglect of non-isothermal problems is the relative complexity of the plasticity theory, which would be further increased by the inclusion of thermal effects.

It is well known that the yield stress of most engineering materials decreases with increase in temperature [1.1]*, [1.2]. As an illustration, we note that for Berryllium the yield stress decreases from 97 x 10^3 psi at room temperature to approximately 20 x 10^3 psi at 1200 F, whereas for Inconel X the corresponding decrease is from 112 x 10^3 psi to 99 x 10^3 psi [1.3]. If the influence of temperature on the yield function and on the stress strain relations is included, it is possible that the stress may decrease rather than increase with the increase of total strain. Such an effect would have no counterpart in the isothermal plasticity theory.

In general, plastic flow and temperature change occur simultaneously and influence each other. The search for the possible forms of non-isothermal plastic stress strain relations is a task of considerable theoretical and practical value.

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* Numbers in brackets refer to the Bibliography.

1-2 Scope of the Investigation

This report is essentially divided into two parts, Part 1 being mainly concerned with the general theory of non-isothermal plasticity. Specifically, in Chapter 2 the First and Second Laws of Thermodynamics, together with the concept of energy conversion, are critically discussed in conjunction with plastic flow. The restriction imposed on the non-isothermal plastic stress strain relations by the Second Law is investigated, and the conditions of loading and unloading for strain hardening and perfectly plastic materials with temperature dependent yield stress are considered in the first part of Chapter 3. Non-isothermal plastic stress strain relations are then presented, and the relations associated with both the Mises and Tresca's yield criteria are shown to satisfy the Second Law of Thermodynamics. Part 11 is devoted to the application of the non-isothermal flow rules developed in Part 1 to illustrative problems. The response of an infinite half space, constrained against lateral motion, to a uniformly applied heat pulse over its boundary is considered. The medium of the half space is assumed to be elastic, linearly strain hardening, and having a yield stress which varies linearly with temperature. Subsequently the cases of elastic, perfectly plastic medium with a constant or temperature dependent yield stress are discussed. Finally, the method of solution is applied to a plate of finite thickness in order to obtain an estimate on the dimensions of the plastically deformed region, since in the half space problem the dimension of length remains arbitrary. One of the faces of the plate is assumed to be subjected to a uniform heat pulse, whereas the other face is either insulated or maintained at zero temperature.

Numerical computations are presented for an aluminum alloy, and illustrate the trends in the variation of transient and residual stresses, strains and displacements. Of particular interest are the dimensions of plastically deformed regions as determined by the maximum amplitudes of the heat pulse.

1-3 Review of Literature

A set of non-isothermal stress strain relations for plasticity was recently proposed by Prager [1.4]; the consistency of these relations with thermodynamic principles has not been investigated, however the work of Prager has been further elaborated by Boley [1.5] and Naghdi [1.6].

Vakulenko [1.7], [1.8] and Ziegler [1.9] have both attempted to formulate the non-isothermal plastic stress strain relations from the point of view of irreversible thermodynamics. A short summary of the basic concepts of this subject may be found in the writings of Freudenthal [1.10], Naghdi [1.11] and Gregarian [1.12]; the fundamentals of irreversible thermodynamics are presented in the well-known texts of Prigogine [1.13], Callen [1.14] and DeGroot [1.15]. Vakulenko and Ziegler start with the assumption that the generalized forces, e.g. stresses, are linearly related to the generalized fluxes, e.g. plastic strain rates, although the validity of this assumption is not immediately obvious. Both authors consider the plastic power as the only quantity associated with the entropy production during non-isothermal plastic deformation; in particular, Ziegler assumed that the stress and plastic strain rate can be derived from two potentials the sum of which is equated to the plastic power. For a strain hardening medium, Vakulenko derived a flow rule from a plastic potential which was found to be a function of the plastic power, the free energy density and another stress potential, but made no reference to the form which the flow rule would assume during unloading and neutral change of state of the medium, nor was there any mention of the relation between a yield function and the flow rule. Although highly interesting, the researches of Vakulenko and Ziegler are still on the nature of exploratory attempts.

Chapter 2 Thermodynamics and Plastic Deformation

2-1 Introduction

The subject of thermodynamics is the study of energy and entropy changes. Specifically, the first law of thermodynamics is concerned with energy balance, whereas the second law characterizes the irreversibility of deformations through the Clausius-Duhem inequality. Any process, reversible or irreversible, is invariably governed by these two laws.

We shall establish in what follows the necessary thermodynamic concepts and develop appropriate forms of the first and second laws of thermodynamics. For convenience, and without loss of generality, we employ rectangular Cartesian coordinates throughout.

2-2 The First Law of Thermodynamics

The First Law of Thermodynamics expresses the condition of energy balance and, specifically, states that the time-rate of change of the total energy in a body is equal to the rate of supply of mechanical and heat energies. We denote the mechanical power of external forces by P and express the flux of heat into the body D as

$$-\int_B q_i\, n_i\, d\sigma = -\int_D q_{i,i}\, d\tau \qquad \text{(a)}$$

where $\bar{q}$ is the heat flux vector, $\bar{n}$ is the outer normal of the boundary B of the region D, and $d\sigma$, $d\tau$ are surface and volume elements respectively.

The First Law may then be stated as follows,

$$P - \int_D q_{i,i}\, d\tau = \dot{K} + \Lambda$$

where $\dot{K}$ is the material time-rate of the kinetic energy K, and Λ is the material time-rate of the total internal energy.

Since

$$K = \frac{1}{2} \int_D \rho v_i\, v_i\, d\tau \quad ,$$

where ρ is the mass density, and v_i the particle velocity, it follows that

$$\dot{K} = \int_D \rho a_i\, v_i\, d\tau \tag{b}$$

Here a_i denotes the particle acceleration defined by

$$a_i = \dot{v}_i$$

The mechanical power P is given by

$$P = \int_B T_i\, v_i\, d\sigma + \int_D \rho f_i\, v_i\, d\tau \tag{c}$$

where T_i and f_i are the components of the surface tractions $\overline{T}$ and body force $\overline{f}$, respectively. The stress tensor τ_{ij} and the stress vector T_i are related by the equation

$$T_i = \tau_{ij}\, n_j \quad ,$$

which, when substituted into (c) yields

$$P = \int_D [(\tau_{ji,j} + \rho f_i)\, v_i + \tau_{ji}\, v_{i,j}]\, d\tau \tag{d}$$

after using the divergence theorem for the surface integral. By virtue of the equation of motion,

$$\tau_{ji,j} + \rho f_i = \rho a_i \tag{e}$$

(d) reduces to

$$\underline{P} = \int_D [\rho a_i \, v_i + \tau_{ij} \, v_{i,j}] \, d\tau \tag{f}$$

Formally, we decompose the velocity gradients $v_{i,j}$ into symmetric and antisymmetric parts,

$$v_{i,j} = d_{ij} + w_{ij}$$

where,

$$d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \; ; \quad w_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i}) \tag{g}$$

are referred to as deformation rate and vorticity, respectively. From (b) and (f) now follows

$$\underline{P} = \dot{k} + \int_D \tau_{ij} \, d_{ij} \, d\tau \tag{h}$$

Since the stress tensor is symmetric, we have that

$$\tau_{ij} \, w_{ij} \equiv 0$$

Therefore, letting

$$\Lambda = \int_D \rho \, \lambda \, d\tau \tag{i}$$

where λ is the rate of internal energy density, we arrive at the form [2.1],[2.2], [2.3], [2.4]

$$\rho\lambda + q_{i,i} = \tau_{ij}\, d_{ij} \qquad (2.2.2)$$

of the law of conservation of energy. It is plausible to write

$$\lambda = \dot{U} + \phi \qquad (j)$$

where $\dot{U}$ is the time rate of a function U of elastic strain ϵ_{ij} and the entropy density s, whereas ϕ represents the time rate of internal energy due to plastic deformation, and is not the time rate of a state function since plastic deformation is a path dependent process. Then

$$\rho\, \dot{U} + \rho\, \phi + q_{i,i} = \tau_{ij}\, d_{ij} \qquad (2.2.3)$$

Equation (2.2.3) is a representation of the principle of balance of mechanical and thermal energies in a form that is most convenient for subsequent investigation.

2-3 Heat Conduction Equation and the Entropy Balance Equation

We assume that the energy U in equation (j) of the preceding section is a function of the elastic strain ϵ_{ij} and the entropy s, thus

$$U = U\,(\epsilon_{ij}\,,\, s) \qquad (2.3.1)$$

We also adopt the customary form of the equations of state [2.5], [2.6],

$$\tau_{ij} = \rho\, \frac{\partial U}{\partial\, \epsilon_{ij}} \quad , \qquad T = \frac{\partial U}{\partial s} \qquad (2.3.2)$$

In the case of small deformations and temperature changes, the function U may be approximated by the first terms of a Taylor's expansion about some reference state. Taking the values of U, τ_{ij}, ε_{ij}, T and s at the reference state to be zero, we obtain

$$U = \frac{1}{2\rho}\left(A\, s^2 + 2b\, s\vartheta + \gamma\vartheta^2 + 2\mu\, \varepsilon_{ij}\, \varepsilon_{ji}\right) \tag{2.3.3}$$

By (2.3.2) and (2.3.3), it follows that

$$T = (A\, s + b\vartheta)/\rho$$

$$\tau_{ij} = \frac{b}{\rho}\, s\, \delta_{ij} + \gamma\vartheta\, \delta_{ij} + 2\Delta\, \varepsilon_{ij}$$

equivalently,

$$s = (T\rho - b\vartheta)/A$$

$$\tau_{ij} = \vartheta\, \delta_{ij}\left(\gamma - \frac{b^2}{A\rho}\right) + 2\mu\, \varepsilon_{ij} + \frac{b}{A}\, \delta_{ij}\, T \tag{2.3.4}$$

where A, b, γ in (2.3.3) and (2.3.4) are known material properties and ϑ is equal to ε_{ii}. We write

$$\left(\gamma - \frac{b^2}{A\rho}\right) = \lambda \tag{a}$$

$$-\frac{b}{A} = (3\lambda + 2\mu)\, \alpha \tag{b}$$

where λ and μ are Lame constants and α is the coefficient of thermal expansion.

Recalling that d_{ij} is the total deformation rate, we have

$$d_{ij} = \overset{\circ}{\varepsilon}_{ij} + s_{ij} \tag{2.3.5}$$

where $\dot{\varepsilon}_{ij}$ and s_{ij} are the elastic and plastic deformation rates, respectively. Substitution of (2.3.2) and (2.3.5) in (2.2.3) then yields:

$$\rho T \dot{s} + q_{i,i} = (\tau_{ij} s_{ij} - \rho \phi) \tag{2.3.6}$$

Eliminating $\overset{\circ}{s}$ in (2.3.6) by the first equation of (2.3.4) and replacing $q_{i,i}$ by $-k T_{,ii}$ through the use of the Fourier's Law of Heat Conduction, we arrive at

$$\rho G(T) \dot{T} - k T_{,ii} = \frac{Tb\dot{\vartheta}}{A} + (\tau_{ij} s_{ij} - \rho \phi) \tag{2.3.7}$$

where $G(T) = T \rho/A$. In particular for small temperature changes and deformation we have that

$$\frac{\rho_o^2 T_o}{A} T - \frac{\rho_o^2 T_o b}{A} \dot{\vartheta} - k T_{,ii} = (\tau_{ij} s_{ij} - \rho \phi)$$

We take

$$\rho_o c = \frac{\rho_o^2 T_o}{A}, \qquad \frac{k}{\rho c} = \kappa \tag{c}$$

where ρ_o T_o are the initial values of the mass density and temperature respectively, c is the specific heat per unit volume and κ is the thermal diffusivity. We now have ~~three~~ conditions, (a), (b) and (c) for explicitly determining A, b and γ.

The expression given by equation (2.3.7) is known as the heat conduction equation. The first term on the right hand side of (2.3.7) may be interpreted as a heat source due to the rate of elastic volume deformation, whereas the term in the parentheses represents another heat source accounting for the dissipation of the inelastic mechanical energy into heat during plastic deformation. Since plastic volume change is usually assumed to be zero, the two heat sources represent two distinctly different phenomenons. The term $Tb\dot{\vartheta}/A$ represents a second-order effect which becomes noticeable only when the duration of motion is very long. Neglecting this term, we write

$$\rho\, G(T)\, \dot{T} - k\, T_{,ii} = (\tau_{ij}\, s_{ij} - \rho\, \phi) \tag{2.3.8}$$

It should be noted that in the dissipation of inelastic mechanical energy, not all of the plastic power $\tau_{ij}\, s_{ij}$ is converted into heat; only the part in excess of the work $\rho\phi$ required to increase the internal energy in conjunction with plastic deformation is dissipated. The function $\rho\phi$ may be considered as the rate at which energy is absorbed by the medium to change its internal structure during plastic deformation.

Dividing (2.3.6) by T, and making use of identity

$$\left(\frac{q_i}{T}\right)_{,i} = \frac{q_{i,i}}{T} - q_i\, \frac{T_{,i}}{T^2} \tag{2.3.9}$$

we obtain the equivalent form:

$$\rho\, \dot{s} + \left(\frac{q_i}{T}\right)_{,i} = -\, q_i\, \frac{T_{,i}}{T^2} + \frac{1}{T}\,(\tau_{ij}\, s_{ij} - \rho\phi) \tag{2.3.10}$$

known as the entropy balance equation.

Let us now introduce an external entropy flux $\dot{S}_e$ by

$$\begin{aligned} \dot{S}_e &= \int_D \rho\, \dot{s}_e \, d\tau = -\int_B \left(\frac{q_i}{T}\right) n_i \, d\sigma \\ &= -\int_D \left(\frac{q_i}{T}\right)_{,i} d\tau \end{aligned} \tag{2.3.11}$$

where $\dot{s}_e$ is the corresponding density. Writing

$$\dot{s} = \dot{s}_e + \dot{s}_i \tag{2.3.12}$$

where $\dot{s}_i$ is defined as the internal entropy production. We note that by (2.3.10) and (2.3.11),

$$\rho\, \dot{s}_i = \rho\, \dot{s} + \left(\frac{q_i}{T}\right)_{,i} \tag{2.3.13}$$

The Second Law of Thermodynamics in the form of the Clausius-Duhem inequality, requires that for any process, the internal entropy production must be non-negative [2.7], [2.8], [2.9].

$$\rho\, \dot{s} + \left(\frac{q_i}{T}\right)_{,i} \geqslant 0 \tag{2.3.14}$$

This implies that the right hand side of (2.3.10) must be greater than zero. Since each term on the right hand side of (2.3.10) represents an independent phenomenon, we require that these terms be separately non-negative, thus

$$-\, q_i \frac{T_{,i}}{T^2} > 0 \tag{2.3.15}$$

and

$$\frac{1}{T}\left(\tau_{ij}\, s_{ij} - \rho\, \phi\right) > 0 \qquad (2.3.16)$$

By Fourier law of heat conduction

$$q_i = -k\, T_{,ii}$$

Equation (2.3.11) therefore requires that

$$k > 0$$

The inequality represented by (2.3.16) must be satisfied in any plastic process. It may be further explored by examining the heat conduction equation, (2.3.8) for two extreme cases. If all the plastic power is absorbed by the material, there is no dissipation of mechanical energy into heat. This would imply that the heat source in (2.3.8) simply vanishes, hence

$$\rho\, \phi = \tau_{ij}\, s_{ij} \qquad (2.3.17)$$

On the other hand, if no energy is being absorbed by the material during plastic deformation, all the plastic power is converted into heat, hence

$$\rho\, \phi = 0 \qquad (2.3.18)$$

In reality these two extreme cases are not likely to obtain [2.10]. It has been demonstrated by Hort [2.11], [2.12], Sato [2.13], Taylor, Farren and Quinney [2.14], [2.15], [2.16] using calorimetric measurements of tension, torsion and compression specimens that not all the plastic power is converted into heat. For example, Taylor and Farren [2.17] found in their experiments that for steel 86.5% of the plastic work was dissipated in the form of heat, hence only 13.5% was absorbed by the material. The percentage of the plastic work converted into heat has been found to range from 90.5 to 92 for aluminum and from 92 to 93 for single aluminum crystal.

We may postulate that

$$\rho \phi = A_e \tau_{ij} s_{ij} \tag{2.3.19}$$

where A_e is a property of the material which can be experimentally determined for a particular material and assumes values between zero and one, i.e.,

$$1 > A_e > 0 \tag{2.3.20}$$

The cases of A_e being equal to zero and one would correspond to the two extreme cases represented by (2.3.17) and (2.3.18). Combining the relations (2.3.16) and (2.3.19), we obtain

$$\frac{1}{T} \tau_{ij} s_{ij} (1 - A_e) > 0 \tag{2.3.21}$$

Since T is always greater than zero, it follows from (2.3.20), that the requirement (2.3.21) is equivalent to

$$\tau_{ij} s_{ij} > 0 \tag{2.3.22}$$

We have therefore shown that, on the basis of the assumption (2.3.19), any flow rule which satisfies relation (2.3.22), will also satisfy the relation (2.3.16) obtained from the Second Law of thermodynamics.

Chapter 3 Non-Isothermal Plastic Stress Strain Relations

3-1 Yield Function and the Criterion of Loading, Unloading and Neutral Change of State

The internal forces of a continuous medium are described by a stress tensor τ_{ij}. Representing the state of stress by a point in six-dimensional space of τ_{ij}, a medium is customarily said to be elastic if the state point lies within a certain convex domain containing the origin of the stress space. This domain is referred to as the elastic domain, and we say that yielding occurs as soon as the state point reaches the boundary of the elastic domain characterized by

$$F\,(\tau_{ij},\ T,\ \xi) = 0 \tag{3.1.1}$$

Here the yield function F is assumed to be a function of stress τ_{ij}, temperature T, and of a strain hardening parameter ξ defined by

$$\xi = \int_0^t (s_{ij}\ s_{ij})^{1/2}\ dt \tag{3.1.2}$$

One of the basic assumptions of plasticity theory is that plastic flow is independent of hydrostatic pressure. Therefore, we shall replace in (3.1.1) the stress τ_{ij} by the deviatoric stress v_{ij} defined by

$$v_{ij} = \tau_{ij} - \frac{1}{3}\ \tau_{kk}\ \delta_{ij} \tag{3.1.3}$$

The equation (3.1.1) now becomes

$$F\,(v_{ij},\ T,\ \xi\ ) = 0 \tag{3.1.4}$$

and we let

$$F(v_{ij}, T, \xi) = f(v_{ij}) - H(T, \xi) \qquad (3.1.5)$$

where $f(v_{ij})$ is a function of deviatoric stress alone and H is a function of temperature T and the strain hardening parameter ξ. Since the medium is assumed to be homogeneous and isotropic, the function f depends on v_{ij} through the invariants

$$J_2(v_{ij}) = \frac{1}{2} v_{ij} v_{ij} = \frac{1}{2}(v_1^2 + v_2^2 + v_3^2)$$
$$(3.1.6)$$
$$J_3(v_{ij}) = \frac{1}{3} v_{ij} v_{jk} v_{ki} = \frac{1}{3}(v_1^3 + v_2^3 + v_3^3)$$

Both functions f and H are further assumed to be continuous and differentiable. We shall adopt as a basic postulate the condition that H increases monotonically with respect to the strain hardening parameter ξ,* i.e.

$$\frac{\partial H}{\partial \xi} > 0 \qquad (3.1.7)$$

Most engineering materials exhibit the characteristic that the yield stress decreases with increasing temperature [3.2], [3.3]. We observe from (3.1.4) and (3.1.5) that during plastic flow the boundary of the elastic domain expands as a result of strain hardening, and that this boundary may also contract or expand depending on the rise or fall of the temperature. If the influence of the temperature on the yield stress dominates, it is probable that with increasing temperature the

*A similar postulate is used by Hill [3.1] for the case when the strain hardening parameter is equal to the total plastic work.

elastic domain may still be shrinking, even if there is strain hardening. Thus the possibility of a decrease in stress accompanied by an increase in strain and temperature in the medium cannot be ruled out. In isothermal plasticity, a medium that exhibits this phenomenon of decrease in stress with increase in strain is considered as unstable since it behaves like an energy source. Furthermore, in isothermal plasticity the decrease in stress automatically implies unloading, which is not always the case in non-isothermal plasticity. It is, therefore, necessary to examine the influence of temperature on the yield function with care, and determine the conditions under which the medium undergoes loading, unloading or neutral change of state.

As is customary in plasticity theory, we set the yield function equal to zero during plastic deformation. If the stress point moves from one plastic state to another in stress space, the plastic strain increases, and we have the conditions

$$F = 0 \quad , \quad \dot{F} = 0 \quad , \quad s_{ij} \lesseqgtr 0 \quad , \quad \dot{\xi} > 0$$

The vanishing of $\dot{F}$ implies

$$\frac{\partial F}{\partial v_{ij}} \dot{v}_{ij} + \frac{\partial F}{\partial T} \dot{T} + \frac{\partial F}{\partial \xi} \dot{\xi} = 0 \qquad (3.1.8)$$

By (3.1.5), and noting that

$$\frac{\partial F}{\partial \xi} = - \frac{\partial H}{\partial \xi} \qquad (3.1.9)$$

The equation (3.1.8) now becomes

$$\frac{\partial F}{\partial v_{ij}} \dot{v}_{ij} + \frac{\partial F}{\partial T} \dot{T} = \frac{\partial H}{\partial \xi} \dot{\xi} \tag{3.1.10}$$

Since $\dot{\xi}$ is a positive definite quantity, as is clear from the definition (3.1.2), and recalling the hypothesis (3.1.7), it follows that

$$\frac{\partial H}{\partial \xi} \dot{\xi} > 0 \tag{3.1.11}$$

The criterion of loading is thus expressed by

$$F = 0 \quad , \quad \dot{F} = 0 \quad , \quad s_{ij} \lesseqgtr 0 \quad \dot{\xi} > 0$$

and

$$\frac{\partial F}{\partial v_{ij}} \dot{v}_{ij} + \frac{\partial F}{\partial T} \dot{T} > 0 \tag{3.1.12}$$

During unloading, the plastic strain rates become zero and the stress point moves from the yield surface back into the elastic domain. The criterion of unloading is expressed by

$$F < 0 \quad \text{or} \quad F = 0 \quad , \quad \dot{F} < 0 \quad , \quad s_{ij} = 0 \quad , \quad \dot{\xi} = 0$$

and

$$\frac{\partial F}{\partial v_{ij}} \dot{v}_{ij} + \frac{\partial F}{\partial T} \dot{T} < 0 \tag{3.1.13}$$

The neutral change of state is then defined by

$$F = 0 \quad , \quad \dot{F} = 0 \quad , \quad s_{ij} = 0 \quad , \quad \dot{\xi} = 0$$

and

$$\frac{\partial F}{\partial v_{ij}} \dot{v}_{ij} + \frac{\partial F}{\partial T} \dot{T} = 0 \tag{3.1.14}$$

For a perfectly plastic material, the yield function F is a function of stress v_{ij} and temperature T alone, i.e.,

$$F(v_{ij}, T) = f(v_{ij}) - H(T) \tag{3.1.15}$$

the criterion of loading, unloading and neutral change of state can be deduced from (3.1.15), (3.1.10), (3.1.12), (3.1.13) and (3.1.14) by setting $\partial H/\partial \xi$ equal to zero. The results are as follows:

<u>Loading</u>

$$F = 0, \quad \dot{F} = 0 \quad , \quad s_{ij} \gtreqless 0 \quad , \quad \dot{\xi} > 0$$

and (3.1.16)

$$\frac{\partial F}{\partial v_{ij}} \dot{v}_{ij} + \frac{\partial F}{\partial T} \dot{T} = 0$$

<u>Unloading</u>

$$F < 0 \quad \text{or} \quad F = 0 \, , \quad \dot{F} < 0 \quad , \quad s_{ij} = 0 \quad , \quad \dot{\xi} = 0$$

and (3.1.17)

$$\frac{\partial F}{\partial v_{ij}} \dot{v}_{ij} + \frac{\partial F}{\partial T} \dot{T} < 0$$

<u>Neutral change of state</u>

$$F = 0 \quad , \quad \dot{F} = 0 \quad , \quad s_{ij} = 0 \quad , \quad \dot{\xi} = 0$$

(3.1.18)

$$\frac{\partial F}{\partial v_{ij}} \dot{v}_{ij} + \frac{\partial F}{\partial T} \dot{T} = 0$$

3-2 Development of Non-Isothermal Plastic Stress Strain Relations

In this section both the strain hardening and perfectly plastic media with temperature dependent yield stress are considered. Since the elastic response is characterized by Hooke's law with an added temperature term, attention is directed mainly to the non-isothermal plastic stress strain relations. It is assumed that an infinitesimal change of the plastic strain r_{ij} is caused by infinitesimal change in the deviatoric stress v_{ij} and the temperature T. Furthermore, the relations between the rates of plastic strain, stress and temperature must be homogeneous of order one so that any results obtained would be indepedent of the time scale employed. We write

$$s_{ij} = A_{ijkl} \dot{v}_{kl} + B_{ij} \dot{T} \tag{3.2.1}$$

where A_{ijkl} and B_{ij} depend on v_{ij}, T and r_{ij}. Incorporating the conditions (3.1.12), (3.1.13), (3.1.14) for loading, unloading and neutral change of state into (3.2.1), the coefficients A_{ijkl} and B_{ij} take on the form

$$A_{ijkl} = C_{ij} \frac{\partial F}{\partial v_{kl}} \quad , \quad B_{ij} = C_{ij} \frac{\partial F}{\partial T} \tag{3.2.2}$$

and the flow law (3.2.1) becomes

$$s_{ij} = \begin{cases} 0 & \text{if} \quad F \leq 0 \ , \ \dot{F} < 0 \\ C_{ij} \left(\frac{\partial F}{\partial v_{kl}} \dot{v}_{kl} + \frac{\partial F}{\partial T} \dot{T} \right) & \text{if} \quad F = 0 \ , \ \dot{F} = 0 \end{cases} \tag{3.2.3}$$

In view of (3.1.12), the quantity in the parenthesis in (3.2.3) is always positive definite during plastic flow. The plastic stress strain relation must further satisfy two necessary conditions [3.4], namely, the condition of plastic incompressibility,

$$s_{ii} = 0$$

and the condition that the direction of principal axes of plastic strain increment coincides with the direction of the principal stress axes, since the medium is assumed to be isotropic. These two conditions are satisfied by choosing

$$c_{ij} = \frac{1}{D} \frac{\partial g}{\partial \tau_{ij}} \tag{3.2.4}$$

where the function g may be considered as a plastic potential, and depends on the invariants J_2 and J_3, whereas D is a function of v_{ij}, T and ξ. The most convenient choice of the function g is to identify it with the function f introduced in (3.1.5). Replacing g by f in (3.2.4) we obtain

$$c_{ij} = \frac{1}{D} \frac{\partial f}{\partial \tau_{ij}} = \frac{1}{D} \left(\frac{\partial f}{\partial J_2} \frac{\partial J_2}{\partial \tau_{ij}} + \frac{\partial f}{\partial J_3} \frac{\partial J_3}{\partial \tau_{ij}} \right) \tag{3.2.5}$$

From the definition (3.1.6) of J_2 and J_3 it is found that

$$\frac{\partial J_2}{\partial \tau_{ij}} = v_{ij} \quad , \quad \frac{\partial J_3}{\partial \tau_{ij}} = v_{jk} \, v_{ki} - \frac{1}{3} \, v_{rk} \, v_{kr} \, \delta_{ij} \tag{3.2.6}$$

From (3.2.6) and (3.2.5) we readily deduce that $C_{ii} = 0$, and thus the condition of plastic incompressibility is met. In terms of the principal rates of stress and strain, the relation (3.2.3) may be written as

$$s_i = \frac{1}{D} \frac{\partial f}{\partial \tau_i} \left(\frac{\partial F}{\partial v_k} \dot{v}_k + \frac{\partial E}{\partial T} \dot{T} \right) \quad \text{for} \quad F = 0, \quad \dot{F} = 0$$

where s_i, τ_i and v_k are principal plastic strain rates, principal total and deviatoric stresses, respectively. The component $\partial f/\partial \tau_i$ represents the projection of the gradient of the function f in the direction of the principal stress τ_i.

It remains to determine the function D. Recalling that during plastic flow both F and $\dot{F}$ are equal to zero and noting that $\dot{\xi}$ is equal to $(s_{ij}\ s_{ij})^{1/2}$, it is found after substituting (3.2.3), (3.2.5) into (3.1.10) that

$$D = \frac{\partial H}{\partial \xi} \left(\frac{\partial f}{\partial \tau_{mn}} \frac{\partial f}{\partial \tau_{mn}} \right)^{1/2} \tag{3.2.7}$$

We recall that $\partial H/\partial \xi$ was postulated to be positive definite during plastic flow, and observe that

$$\left(\frac{\partial f}{\partial \tau_{mn}} \quad \frac{\partial f}{\partial \tau_{mn}} \right)^{1/2}$$

is positive definite if the positive root of the quantity in the parenthesis is taken. Therefore the quantity D is positive definite.

Combining (3.2.3), (3.2.5) and (3.2.7), the non-isothermal plastic flow law now becomes

$$s_{ij} = \begin{cases} 0 & \text{for } F \le 0,\ \dot{F} < 0 \\ \dfrac{\dfrac{\partial f}{\partial \tau_{ij}}}{\dfrac{\partial H}{\partial \xi}\left(\dfrac{\partial f}{\partial \tau_{mn}}\ \dfrac{\partial f}{\partial \tau_{mn}}\right)^{1/2}}\left(\dfrac{\partial F}{\partial v_{kl}}\ \dot{v}_{kl} + \dfrac{\partial F}{\partial T}\dot{T}\right) & \text{for } F = 0,\ \dot{F} = 0 \end{cases} \tag{3.2.8}$$

The formula (3.2.8) is a general flow rule for strain hardening media with temperature dependent yield stress, and can be used to determine the plastic strain rates once the function F is known.

The two most commonly used yield criteria are due to Von Mises and Tresca respectively. The yield function defined by the Von Mises yield criterion is given by [3.5], [3.6]

$$\begin{aligned} F &= f - H \\ &= J_2 - K^2\,(T,\xi) \end{aligned} \tag{3.2.9}$$

where

$$f = J_2 \quad , \quad H = K^2\,(T,\xi)$$

and K is the yield stress in simple shear. The yield function represented by (3.2.9) is said to be regular because a unique normal can be defined at every point on the yield surface. If (3.2.9) is substituted into (3.2.8), the flow rule associated with the Von Mises yield criterion then takes on the form

$$s_{ij} = \begin{cases} 0 & \text{for } F \le 0,\ \dot{F} < 0 \\ \dfrac{v_{ij}}{\dfrac{\partial H}{\partial \xi}\left(v_{mn}\ \ v_{mn}\right)^{1/2}}\left(\dfrac{\partial F}{\partial v_{kl}}\ \dot{v}_{kl} + \dfrac{\partial F}{\partial T}\dot{T}\right) & \text{for } F = 0,\ \dot{F} = 0 \end{cases} \tag{3.2.10}$$

The flow rule for a perfectly plastic medium with temperature dependent yield stress can be established by considering again the condition of the coincidence of the direction of principal axes of plastic strain rates with the principal stress axes during plastic flow, and the condition of plastic incompressibility. These conditions suggest the flow rule

$$s_{ij} = \begin{cases} 0 & \text{for } F \leq 0 \;, \quad \dot{F} < 0 \\ \lambda \dfrac{\partial f}{\partial \tau_{ij}} & \text{for } F = 0 \;, \quad \dot{F} = 0 \end{cases} \tag{3.2.11}$$

where λ is a function of plastic strain, stress and temperature and their rates. It is not possible to determine λ from the yield function itself because the latter is independent of plastic strain. If we consider the yield criterion of Von Mises,

$$\begin{aligned} F &= f - H \\ &= J_2 - K^2(T) \end{aligned} \tag{3.2.12}$$

then λ may be obtained by multiplying (3.2.11) by τ_{ij} and summing. In view of the definition (3.1.6) of J_2, we find that

$$\lambda = \frac{\tau_{ij}\, s_{ij}}{2K^2(T)} \tag{3.2.13}$$

3-3 Non-Isothermal Plastic Stress Strain Relations for Singular Yield Functions

The yield function defined by the Von Mises yield criterion is a regular yield function representing a yield surface without corners. We may also characterise yielding by a set of n yield functions $F^{(\gamma)}(v_{ij}, T, \xi)$, where $\gamma = 1,2,\ldots n$. These yield functions define a yield surface in the stress space which may not have a unique normal at every point, and are referred to as singular yield functions. We adopt the convention that plastic flow begins whenever any one of the functions $F^{(\gamma)}$ becomes zero, and that the medium is elastic if all functions $F^{(\gamma)}$ are less than zero. The plastic strain rate is then taken to be the sum of the constituents $s_{ij}^{(\gamma)}$ [3.7], [3.8], [3.9], thus

$$s_{ij} = \sum_{\gamma=1}^{n} s_{ij}^{(\gamma)} \tag{3.3.1}$$

For a strain hardening medium with a temperature dependent yield stress, the plastic strain rates $s_{ij}^{(\gamma)}$ are defined by

$$\begin{aligned}
& s_{ij}^{(\gamma)} = 0 \;, \quad \dot{\xi}^{(\gamma)} = 0 \;, \quad F^{(\gamma)} < 0 \quad \text{or} \quad F^{(\gamma)} = 0 \;, \; \dot{F}^{(\gamma)} < 0 \\
& \frac{\partial F^{(\gamma)}}{\partial v_{ij}} \dot{v}_{ij} + \frac{\partial F^{(\gamma)}}{\partial T} \dot{T} < 0 \\
& s_{ij}^{(\gamma)} \gtreqless 0, \quad \dot{\xi} > 0 \;, \quad F^{(\gamma)} = 0 \;, \quad \dot{F}^{(\gamma)} = 0 \\
& \frac{\partial F^{(\gamma)}}{\partial v_{ij}} \dot{v}_{ij} + \frac{\partial F^{(\gamma)}}{\partial T} \dot{T} > 0
\end{aligned} \tag{3.3.2}$$

where $s_{ij}^{(\gamma)}$ and $\xi^{(\gamma)}$ are the values of s_{ij} and ξ corresponding to the function $F^{(\gamma)}$. The most commonly used singular yield functions due to Tresca, and are defined by [3.10], [3.11]

$$F^{(1)} = f^{(1)} - H(T,\xi)$$
$$F^{(2)} = f^{(2)} - H(T,\xi) \qquad (3.3.3)$$
$$F^{(3)} = f^{(3)} - H(T,\xi)$$

where

$$f^{(1)} = \frac{|v_1 - v_2|}{2} = \frac{|\tau_1 - \tau_2|}{2}$$
$$f^{(2)} = \frac{|v_2 - v_3|}{2} = \frac{|\tau_2 - \tau_3|}{2} \qquad (3.3.4)$$
$$f^{(3)} = \frac{|v_3 - v_1|}{2} = \frac{|\tau_3 - \tau_1|}{2}$$

and v_i, τ_i are the principal values of the deviatoric and total stress tensors. The function H is identified with the yield stress; for a perfectly plastic medium H is a function of temperature T alone.

The plastic strain rates $s_{ij}^{(\gamma)}$ in (3.3.1) and (3.3.2) are obtained from (3.2.8) by replacing the functions f, F and H by the corresponding functions $f^{(\gamma)}$, $F^{(\gamma)}$ and H associated with the singular yield surface.

We note that, for Tresca's yield criterion not more than two of the three singular yield functions may vanish [3.12] , the third one would be equal to -H. In this case the stress point is at the corner of the yield surface defined by (3.3.3) in the space of principal stresses. We remark that Tresca's yield criterion is frequently used when the direction and sense of the principal stresses are known from consideration of symmetry.

From (3.2.11) follows that, for a perfectly plastic medium, the flow rule associated with the Tresca's yield criterion is expressed by

$$s_i = \sum_{\gamma=1}^{3} \lambda^{(\gamma)} \frac{\partial f^{(\gamma)}}{\partial \tau_i} \qquad (3.3.5)$$

where

$$\lambda^{(\gamma)} = 0 \quad \text{if} \quad F^{(\gamma)} < 0 \quad \text{or} \quad F^{(\gamma)} = 0\,, \quad \dot{F}^{(\gamma)} < 0$$

$$\frac{\partial F^{(\gamma)}}{\partial v_k} \dot{v}_k + \frac{\partial F^{(\gamma)}}{\partial T} \dot{T} < 0$$

and (3.3.6)

$$\lambda^{(\gamma)} > 0 \quad \text{if} \quad F^{(\gamma)} = 0 \quad , \quad \dot{F}^{(\gamma)} = 0$$

$$\frac{\partial F^{(\gamma)}}{\partial v_k} \dot{v}_k + \frac{\partial F}{\partial T} \dot{T} = 0$$

The quantities $\lambda^{(\gamma)}$ are functions of stress, temperature, strain and their rates. These coefficients may be determined in a manner similar to that used in the last section, although for particular problems other methods may be more suitable (cf. chapter 5).

3-4 Thermodynamic Investigation of the Flow Rule

The conditions imposed on flow rules by the principle of thermodynamics were derived in the preceding chapter. Specifically, we found that non-negative entropy production requires that

$$\tau_{ij}\, s_{ij} > 0 \qquad (2.3.22)\text{R}$$

be valid. We intend to show in this section that the general non-isothermal flow rule given by (3.2.8) satisfies (2.3.22), and, therefore, the second law of thermodynamics, when used in conjunction with either the Von Mises or Tresca's yield criterion.

We recall the flow rule (3.2.10) associated with yield criterion of Von Mises, and, multiplying it by τ_{ij} and summing, obtain the result

$$\tau_{ij}\, s_{ij} = \frac{\tau_{ij}\, v_{ij}}{\frac{\partial H}{\partial \xi}\,(v_{mn}\, v_{mn})^{1/2}} \left(\frac{\partial F}{\partial v_{kl}}\, \dot{v}_{kl} + \frac{\partial F}{\partial T}\, \dot{T}\right) \quad (3.4.1)$$

In order to verify (2.3.22), we first recall

$$\left(\frac{\partial F}{\partial v_{kl}}\, \dot{v}_{kl} + \frac{\partial F}{\partial T}\, \dot{T}\right) > 0 \qquad (3.1.12)R$$

and that $\partial H/\partial \xi$ was also assumed to be positive (cf. relation (3.1.7)). Then, since $(v_{mn}\, v_{mn})^{1/2}$ is clearly positive, the only quantity that needs to be examined is $\tau_{ij}\, v_{ij}$.

Writing

$$\tau_{ij} = v_{ij} + \frac{1}{3}\, \delta_{ij}\, \tau_{kk} \qquad (3.1.3)R$$

we find that

$$\tau_{ij}\, v_{ij} = v_{ij}\, v_{ij} + \frac{1}{3}\, \delta_{ij}\, v_{ij}\, \tau_{kk}$$

$$= v_{ij}\, v_{ij} \geq 0 \qquad (3.4.2)$$

where we used the fact that, be definition

$$v_{ii} = 0$$

i.e., the first invariant of the deviatoric stress tensor vanishes identically. The result (3.4.2) thus completes the proof of (2.3.22) for the case when the yield criterion of Von Mises is adopted.

In order to verify (2.3.22) for the flow rule (3.2.8) used in conjunction with the Tresca's yield criterion, we consider a stress point at a corner of the yield surface. The case when only one singular function vanishes is then included as a special case. Let us assume that in the yield corner

$$v_1 > v_2 \quad , \quad v_1 > v_3 \tag{3.4.3}$$

and that the singular yield functions are now represented by

$$\begin{aligned} F^{(1)} &= \frac{v_1 - v_2}{2} - H = 0 \\ F^{(2)} &= \frac{v_1 - v_3}{2} - H = 0 \\ F^{(3)} &= \frac{|v_2 - v_3|}{2} - H < 0 \end{aligned} \tag{3.4.4}$$

In addition, we let

$$\dot{F}^{(1)} = 0 \quad , \quad \dot{F}^{(2)} = 0 \quad , \quad \dot{F}^{(3)} < 0 \tag{3.4.5}$$

Noting that

$$v_1 - v_2 = \tau_1 - \tau_2 \quad , \quad v_1 - v_3 = \tau_1 - \tau_3$$

the assumption (3.4.3) implies

$$\tau_1 - \tau_2 > 0 \quad , \quad \tau_1 - \tau_3 > 0 \tag{3.4.6}$$

The first two equations of (3.4.4) can be written as

$$f^{(1)} = \frac{v_1 - v_2}{2} = \frac{\tau_1 - \tau_2}{2} = H$$
$$f^{(2)} = \frac{v_1 - v_3}{2} = \frac{\tau_1 - \tau_3}{2} = H \qquad (3.4.7)$$

Since the function H is the same for all three singular yield functions, this implies that

$$v_2 = v_3 \quad , \quad \tau_2 = \tau_3$$
$$f^{(1)} = f^{(2)} = f^c, \quad F^{(1)} = F^{(2)} = F^c \qquad (3.4.8)$$

where f^c and F^c are the values of the functions $f^{(1)}$, $f^{(2)}$ and $F^{(1)}$, $F^{(2)}$ at the yield corner, respectively.

Multiplying (3.3.1) by τ_{ij}, and noting that

$$\tau_{ij}\, s_{ij} = \tau_k\, s_k$$

we obtain

$$\tau_k\, s_k = \tau_k \sum_{\gamma=1}^{3} s_k^{(\gamma)} \qquad (3.4.9)$$

Since, by (3.2.8),

$$s_i^{(\gamma)} = \frac{\dfrac{\partial f^{(\gamma)}}{\partial \tau_i}}{\dfrac{\partial H}{\partial \xi}\left(\dfrac{\partial f^{(\gamma)}}{\partial \tau_{mn}}\,\dfrac{\partial f^{(\gamma)}}{\partial \tau_{mn}}\right)^{1/2}}\left(\frac{\partial F^{(\gamma)}}{\partial v_n}\,\dot{v}_n + \frac{\partial F^{(\gamma)}}{\partial T}\,\dot{T}\right) \qquad (3.4.10)$$

the substitution of (3.4.10) into (3.4.9) with (3.4.5), (3.4.7) and (3.4.8) yields

$$\tau_1 s_1 + \tau_2 s_2 + \tau_3 s_3$$

$$= [\tau_1 \ (\frac{\partial f^{(1)}}{\partial \tau_1} + \frac{\partial f^{(2)}}{\partial \tau_1}) + \tau_2 \ (\frac{\partial f^{(1)}}{\partial \tau_2}) + \tau_3 \ (\frac{\partial f^{(2)}}{\partial \tau_3})]$$

$$\cdot \ \frac{1}{\frac{\partial H}{\partial \xi} \ (\frac{\partial f^c}{\partial \tau_m} \frac{\partial f^c}{\partial \tau_m})^{1/2}} \ (\frac{\partial F^c}{\partial v_n} \ \dot{v}_n + \frac{\partial F^c}{\partial T} \ \dot{T}) \qquad (3.4.11)$$

The flow rule associated with the Tresca's yield criterion is compatible with the second law of thermodynamics if the right hand side of (3.4.11) can be shown to be greater than zero. The quantities $(\partial H/\partial \xi)$, $(\partial f^c/\partial \tau_m \ \partial f^c/\partial \tau_m)^{1/2}$ and $(\partial F^c/\partial v_n \ \dot{v}_n + \partial F^c/\partial T \ \dot{T})$ are all positive definite during plastic flow. The only quantity that needs to be examined is the sum of the terms inside the square bracket. By (3.4.7) and (3.4.8), we obtain

$$\frac{\partial f^{(1)}}{\partial \tau_1} = \frac{\partial f^{(2)}}{\partial \tau_1} = \frac{1}{2} \ , \ \frac{\partial f^{(1)}}{\partial \tau_2} = - \frac{1}{2} \ , \ \frac{\partial f^{(2)}}{\partial \tau_3} = - \frac{1}{2}$$

(3.4.12)

Hence, in view of (3.4.6), we have that

$$\left[\tau_1 \left(\frac{\partial f^{(1)}}{\partial \tau_1} + \frac{\partial f^{(2)}}{\partial \tau_1} \right) + \tau_2 \left(\frac{\partial f^{(1)}}{\partial \tau_2} \right) + \tau_3 \left(\frac{\partial f^{(2)}}{\partial \tau_3} \right) \right]$$

$$= \left[\frac{1}{2} (\tau_1 - \tau_2) + \frac{1}{2} (\tau_1 - \tau_3) \right] > 0$$

Thus it has been shown that the flow rule used on conjunction with Tresca's yield criterion is also compatible with the Second Law of Thermodynamics.

3-5 <u>A Special Case of the General Flow Rule</u>

We now proceed to exhibit a further justification of the form (3.2.8) of the flow rule. Specifically, in this section it will be shown that the Von Mises flow rule formulated for perfectly plastic media is a special case of the general flow rule.

By (3.1.10), the general flow rule can be written as

$$s_{ij} = \frac{\dfrac{\partial f}{\partial \tau_{ij}}}{\dfrac{\partial H}{\partial \xi} \left(\dfrac{\partial f}{\partial \tau_{mn}} \dfrac{\partial f}{\partial \tau_{mn}} \right)^{1/2}} \left(\frac{\partial H}{\partial \xi} \dot{\xi} \right) \tag{3.5.1}$$

With the yield function of Von Mises represented by

$$F = \frac{1}{2} v_{mn} v_{mn} - K^2 \tag{3.5.2}$$

where K is the constant yield stress in simple shear, the flow rule now becomes

$$s_{ij} = \frac{v_{ij}}{(v_{mn} v_{mn})^{1/2}} \dot{\xi} \tag{3.5.3}$$

By definition (3.1.2), $\dot{\xi}$ is equal to $(s_{kl}\, s_{kl})^{1/2}$. After re-arranging, (3.5.3) now appears as

$$v_{ij} = \frac{(v_{mn}\, v_{mn})^{1/2}}{(s_{kl}\, s_{kl})^{1/2}} \quad s_{ij} \tag{3.5.4}$$

From (3.5.2) follows that, during plastic flow, $(v_{mn}\, v_{mn})^{1/2}$ is equal to $K\sqrt{2}$, therefore

$$\left(\frac{v_{mn}\, v_{mn}}{s_{kl}\, s_{kl}} \right)^{1/2} = \frac{K}{(\frac{1}{2}\; s_{kl}\, s_{kl})^{1/2}} = \frac{K}{\sqrt{I}} \tag{3.5.5}$$

where

$$I = \frac{1}{2}\, s_{kl}\, s_{kl}$$

Combining (3.5.5) and (3.5.4), we arrive at

$$v_{ij} = \frac{K}{\sqrt{I}} \quad s_{ij} \tag{3.5.6}$$

The expression given by (3.5.6) is the exact form of the Mises flow rule for perfectly plastic media [3.13]. This implies that the general flow rule as represented by (3.2.8), under certain assumption can be reduced to some known plastic stress strain relation, e.g. the Mises flow rule.

PART II. APPLICATION

Chapter 4. The Elastic-Plastic Response of a Half Space to a Uniformly Applied Heat Pulse at its Boundary

4-1 Introduction

Temperature variations of large amplitude and short duration are encountered in various fields of engineering applications. For example, the elements of a nuclear reactor may be subjected to sudden heating and cooling due to changes in the rate of fission. Similarly, a spacecraft is subjected to heat pulses associated with the ignition of rockets, and with re-entry into the atmosphere of the earth [4.1], [4.2], [4.3]. It is naturally important to be able to predict the transient and residual stresses and deformations caused by such heat pulses and to investigate the physical damage incurred.

Transient thermal stress and deformation in the elastic-plastic range have been recently studied by a number of investigators. In particular, various problems for plates have been treated by Weiner [4.4], Yuksel [4.5], Landau and associates [4.6], and most recently by Mendelson and Spero [4.7]. Except for the work of Mendelson, the incremental theory has been used in every case, and the medium considered was assumed to be elastic, perfectly plastic obeying the Von Mises yield condition and possessing a constant yield stress. Landau [4.8], however, in one case did take the dependence of the yield stress on temperature into consideration. Mendelson employs the deformation theory without resorting to any flow rule of yield condition, but takes strain hardening and dependence of the physical properties of the medium upon temperature into account. Similar investigations pertaining to elastic, perfectly plastic cylinders were considered by Weiner and Huddleston [4.9], Landau and Zwicky [4.10]. In the latter investigation the yield condition of Von Mises is adopted, and the yield stress is assumed to depend on the temperature. A similar problem pertaining to elastic, strain hardening spheres with a constant yield stress was treated by Huang [4.11]; because of the complexity of the numerical calculations in the work of Huang and Landau, high speed computers were used.

In order to illustrate the use of the concepts developed in the first part of this study, we shall present a detailed analysis of the transient and residual stresses and deformations induced in a half space by a heat pulse uniformly distributed over its boundary. The half space is assumed to be constrained against lateral motion, and the medium is assumed to be elastic, homogeneous and isotropic, and linearly strain hardening in the plastic range. Moreover, the yield stress is permitted to vary linearly with temperature, whereas other material properties are assumed constant.

We shall give a complete study of the growth and decay of various regions of loading and unloading. The most serious limitation of the solution to this problem is that it lacks a characteristic dimension of length. Therefore, we shall briefly investigate a plate of finite thickness in Chapter 6.

4-2 The Temperature Problem

As a consequence of the mathematical complexity inherent in the thermal stress problems, it is necessary to select temperature solutions that are as simple as possible. In particular, the mathematical problems are greatly simplified if the temperature solution is in a closed form. One class of such closed form solutions corresponds to temperature fields induced by instantaneous heat sources (or sinks), doublets, and the like. Since these solutions are singular at the source point, the latter must be located outside the material body.

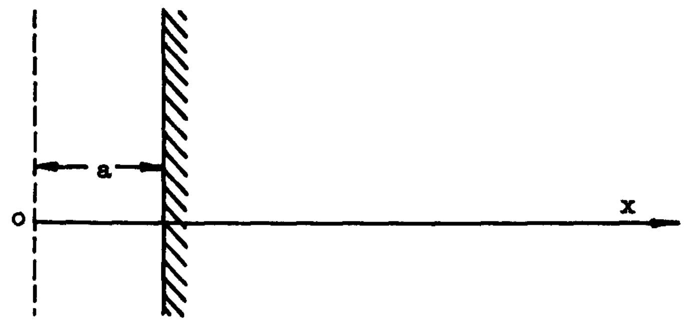

Fig. 4-1 Heat Sources and the Infinite Slab

In the present analysis, a row of heat sources is placed at an arbitrary distance a from the boundary of the body (Fig. 4-1). Let x be the coordinate measured from the source plane, and t the time. The increase of temperature T over some reference temperature T_o is then given by [4.12]

$$T = C \frac{e^{-x^2/4\kappa t}}{t^{1/2}} \tag{4.2.1}$$

where C is a constant, and κ is the diffusivity of the material (assumed to be constant). It may be readily verified that the temperature field (4.2.1) satisifes the one-dimensional heat conduction equation

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\kappa} \frac{\partial T}{\partial t} \tag{4.2.2}$$

and is finite everywhere in the material (i.e. $x \geqslant a$).

The maximum temperature T_m occurs at the boundary $x = a$, and, as may be found from (4.2.1), at the instant t_m given by

$$t_m = \frac{a^2}{2\kappa} \tag{4.2.3}$$

Similarly it is found that the maximum temperature at x is reached at the instant t_{mx} given by

$$t_{mx} = \frac{x^2}{2\kappa} \tag{4.2.4}$$

The arbitrary constant C in (4.2.1) may now be eliminated in terms of a given maximum temperature T_m, thus obtaining

$$T = T_m \frac{a}{(2 \kappa t)^{1/2}} \quad e^{\frac{1}{2} - x^2/4\kappa t} \tag{4.2.5}$$

Substituting (4.2.4) into (4.2.5), the maximum temperature T_{mx} attained at x becomes

$$T_{mx} = T_m \frac{a}{x} \tag{4.2.6}$$

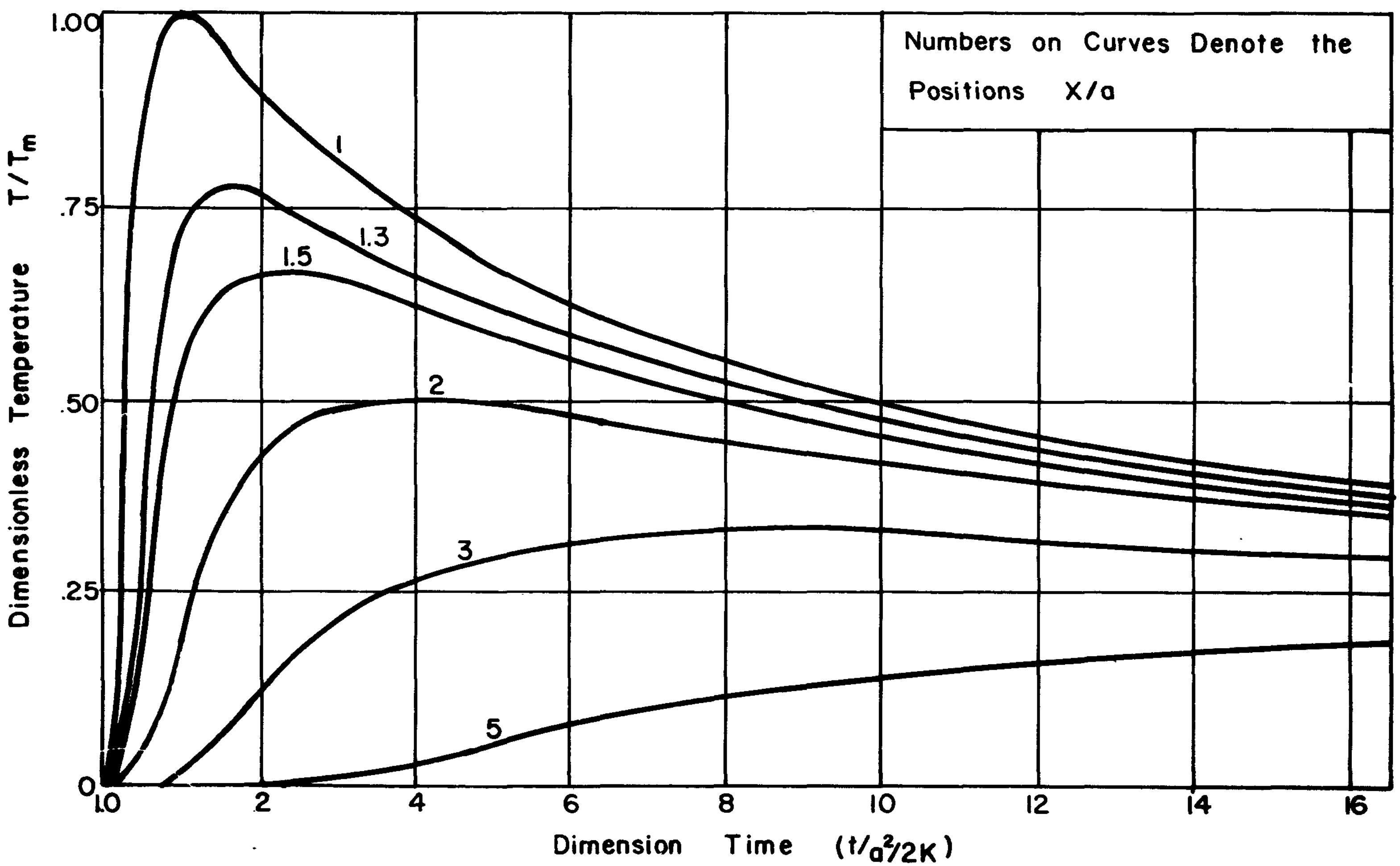

FIG 4-2 TEMPERATURE VARIATION WITH TIME AT DIFFERENT POSITIONS : (4.25)

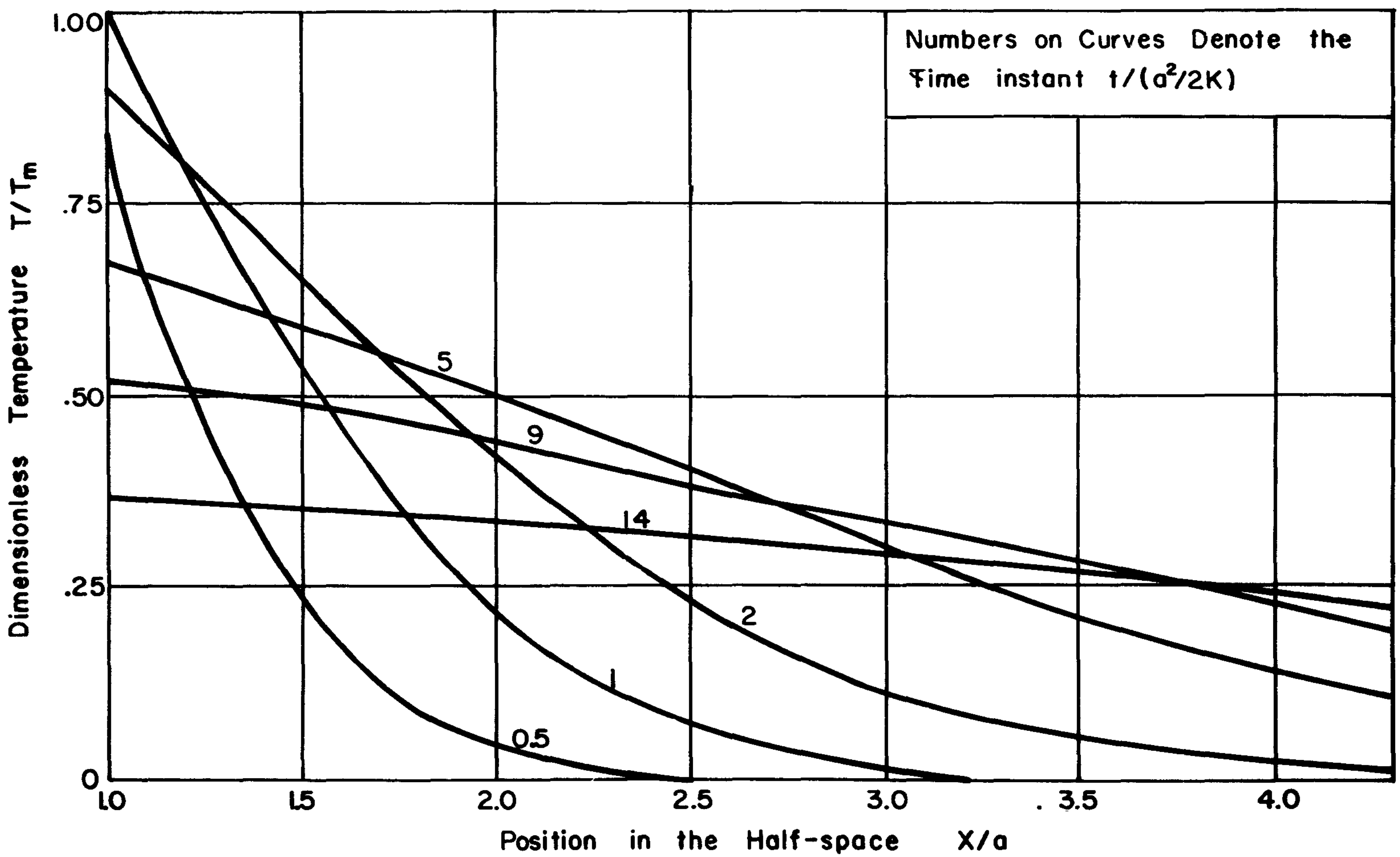

FIG. 4-3 TEMPERATURE DISTRIBUTION AT DIFFERENT INSTANTS OF TIME

The variation of the temperature T with time t for various values of position x is shown in Fig. 4-2, whereas Fig. 4-3 illustrates the variation of T with x for various values of t. The results have been made dimensionless by using the variables

$$T/T_m, \quad x/a, \quad t/t_m$$

for temperature, position and time respectively.

The solution (4.2.1) corresponds to a temperature field induced by suddenly releasing a finite amount of heat at $x = 0$. It may be seen from Fig. 4-2 that this solution is a reasonable approximation of a heat shock, i.e., of a sudden rise in the boundary temperature followed by a gradual decay. It has been assumed, of course, that the heat conduction problem is not influenced by the mechanical deformation of the material.

4-3 The Elastic Regime

During the initial stages of the heat shock, the response of the material will be elastic. Let τ_{ij}, ϵ_{ij}, u_i denote stresses, strains and displacements respectively. The elastic thermal stress problem is then characterized by the stress equation of equilibrium

$$\tau_{ij,j} = 0 \tag{4.3.1}$$

The stress strain relations

$$E_1 \epsilon_{ij} - E_1 \alpha T \delta_{ij} = \tau_{ij} - (1+\nu_1) \tau_{kk} \delta_{ij} \tag{4.3.2}$$

and the strain displacement relations

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{4.3.3}$$

where E_1, α, ν_1 are Young's modulus, coefficient of thermal expansion and Poisson's ratio, respectively, and use has been made of the conventional index notation.

In addition to quiescent initial conditions, it is assumed that at the boundary

$$\tau_{xx} = 0 \qquad \text{for} \qquad x = a \tag{4.3.4}$$

In view of the simplicity of the problem, the following abbreviated notation is introduced,

$$U_x = U(x,t) \quad , \qquad U_y = U_z = 0 \tag{4.3.5}$$

$$\epsilon_{xx} = \epsilon(x,t) \quad , \qquad \epsilon_{yy} = \epsilon_{zz} = 0 \tag{4.3.6}$$

$$\tau_{yy} = \tau_{zz} = \tau(x,t), \qquad \tau_{xx} = 0 \tag{4.3.7}$$

The relations (4.3.5) express the condition that the half-space is constrained against lateral motion. It follows, then, that all shearing stresses and shearing strains vanish. Finally, the second of (4.3.7) follows from (4.3.1) and (4.3.4).

Taking (4.3.5), (4.3.6), (4.3.7) into account, the relation (4.3.2) finally appears as

$$E_1 \epsilon - E_1 \alpha T = -2 \nu_1 \tau \quad ,$$

$$-\alpha E_1 T = \tau (1 - \nu_1) \quad ,$$

whence follows

$$\epsilon = \epsilon_{xx} = \frac{1 + \nu_1}{1 - \nu_1} \alpha T \quad , \tag{4.3.8}$$

$$\tau = \tau_{yy} = \tau_{zz} = - \frac{\alpha E_1}{1 - \nu_1} T \quad , \tag{4.3.9}$$

Here T is given by (4.2.5), and we note that

$$\tau = 2 \mu_E \epsilon \tag{4.3.10}$$

where

$$\mu_E = \frac{E_1}{2(1 + \nu_1)} \tag{4.3.11}$$

is the elastic modulus of shear. Introducing the maximum shearing stress q

$$q = -\frac{\tau}{2} = \frac{\alpha E_1}{2(1 - \nu_1)} T \tag{4.3.12}$$

and the maximum shearing strain p

$$p = \frac{\epsilon}{2} = \frac{1 + \nu_1}{2 (1 - \nu_1)} \alpha T \quad , \tag{4.3.13}$$

the relation (4.3.10) may now be written as

$$q = 2 \mu_E p \tag{4.3.14}$$

Finally, integrating (4.3.8), the displacement U is obtained as

$$U = \frac{1 + \nu_1}{1 - \nu_1} \, a \, \alpha \, T_m \, (e \frac{\pi}{2})^{1/2} \left[\psi \left(\frac{x}{2 \sqrt{\kappa t}} \right) - 1 \right] \tag{4.3.15}$$

where

$$\psi(s) = \frac{2}{(\pi)^{1/2}} \int_0^s e^{-u^2} \, d u$$

is the error function, and the arbitrary function of time has been chosen so that the regularity condition

$$U = 0 \qquad \text{for} \qquad x \to \infty \tag{4.3.16}$$

is satisfied.

We also make the relevant observation that the solution given by (4.3.8), (4.3.9), (4.3.15) holds even if the elastic region begins at some elastic-plastic interface characterized by $x = \rho_1$.

4-4 The Elastic-Plastic Regime

4-41 The Onset of Yielding

It is found from the elastic solution presented in the preceding section that, as the temperature increases, the shearing stress also increases. Let the initial yield stress in shear be denoted by y_I, and let

$$y_I = y_o - BT \tag{4.4.1}$$

where B is assumed to be a known constant, and y_o is the value of y_I for $T = 0$ (i.e. at the reference temperature T_o). The instant t_1 at which yielding begins at the boundary $x = a$ can be readily determined by letting

$$q\,(a,t) = y_I\,(a,t) \quad , \tag{a}$$

if we adopt the yield condition of Tresca. From (4.3.12), (4.4.1) and (4.2.5) we then obtain

$$\log t_1 + \frac{a^2}{2\,\kappa\, t_1} = -\,2 \log \left\{ (2\kappa/e)^{1/2} \; y_o/a\, T_m \left[B + \frac{E_1\,\alpha}{2(1-\nu_1)} \right] \right\} \tag{4.4.2}$$

The equation (4.4.2) can also be used to find the instant t_{1x} at which yielding occurs at the plane characterized by x, if a and t_1 are replaced there by x and t_{1x}, respectively.

From (4.3.12), (4.4.1) and (a) follows that the boundary temperature T_1 at the instant t_1 is given by

$$T_1 = \frac{y_o}{B + E_1\,\alpha/2(1 - \nu_1)} \tag{4.4.3}$$

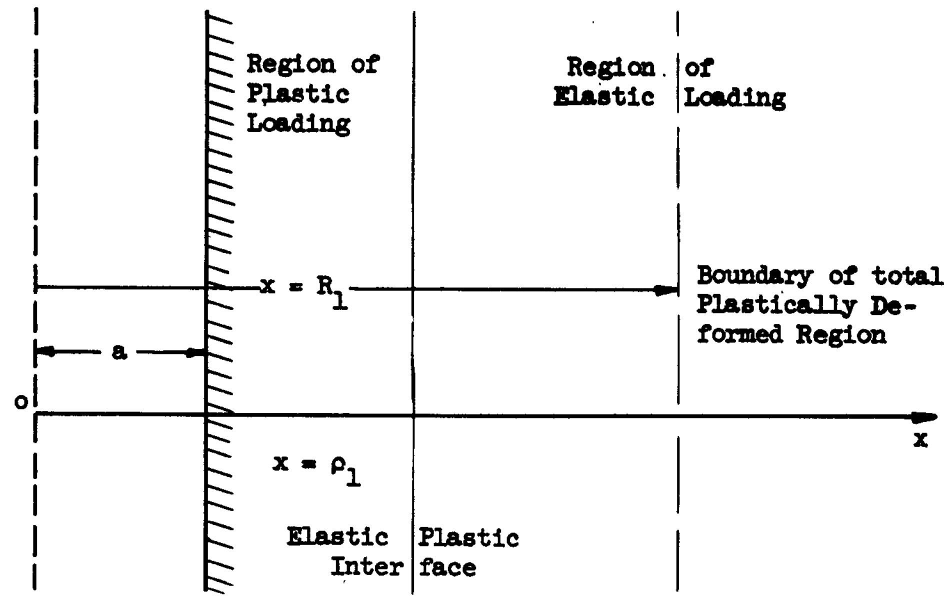

Fig. 4-4 Regions of Loading During the Elastic-Plastic Regime

For $t > t_1$ an elastic plastic interface whose location is denoted by ρ_1, progresses into the material. In the wake of this interface, the material undergoes plastic loading until t becomes equal to t_m; then, corresponding to the decrease of the boundary temperature, an elastic unloading front emanates from the boundary.

The present section concerns the time interval $t_1 \leq t \leq t_m$, hence, in addition to the solution of the elastic region, it is necessary to derive a solution for the plastic region.

4-42 <u>The Elastic Region</u>

The solution of the elastic region is the same as that given in section 4-3 except that the elastic region now lies at $x > \rho_1$, where ρ_1 is yet to be determined.

In order to determine the position of the elastic-plastic interface ρ_1, we use (4.2.5), (4.4.1), (4.3.12) and (4.3.9). In particular, from

$$q\,(\rho_1,\, t) = y_I\,(\rho_1, t) \tag{4.4.4}$$

follows

$$\frac{1}{2}\,\frac{\alpha\, E_1}{1-\nu_1}\, T\,(\rho_1, t) = y_o - B\,T\,(\rho_1, t)$$

or

$$\rho_1^{\,2} = 2\,\kappa\, t \left\{ 1 - \log t - 2\log\left[y_o\,(2k)^{1/2} / T_m\, a\, \left(\frac{E_1\,\alpha}{2(1-\nu_1)} + B\right)\right]\right\} \tag{4.4.5}$$

It is interesting to note that, as a consequence of (4.4.4), the temperature at the interface remains constant and equal to T_1 (the boundary temperature corresponding to incipient yielding), thus

$$T(\rho_1\ t) = T_{1x} = T_1 = \frac{y_o}{B + E_1\,\alpha/2(1-\nu_1)} \tag{4.4.6}$$

where T_{1x} is the temperature corresponding to incipient yielding at the plane characterized by x.

It follows then, that the progress of the interface may be studied by drawing on Fig. 4-2 a horizontal line at the height of T_1/T_m, and investigating its intercepts with temperature curves.

Since T_1 is the maximum temperature for yielding, the maximum value R_1 of ρ_1 can be determined easily. Namely, since the maximum temperature T_{mx} reached at x is given by

$$T_{mx} = \frac{a}{x} T_m \tag{4.2.6)R}$$

it follows that

$$T_1 = \frac{a}{R_1} T_m$$

or by (4.4.6)

$$R_1 = a \frac{T_m}{y_o} \left[B + \frac{E_1 \alpha}{2(1-\nu_1)} \right] \tag{4.4.7}$$

Therefore, it is known that the region $a \leq x \leq R_1$ will undergo some plastic deformation. The exact determination of the residual state, however, can only be found from a detailed analysis.

4-43 <u>The Plastic Region of Loading</u>

As the boundary temperature rises beyond T_1, the plastic region extends inward from the boundary. In order to derive the appropriate solution for the plastic region, the plastic stress strain relations must be established first; this step must be preceded by the formulation of yield functions, however.

4-43-1 <u>The Yield Functions</u>

In the elastic solution the maximum shearing stress q and maximum shearing strain p is related by

$$q = 2\mu_E p \qquad (4.3.14)R$$

Since the medium is assumed to be linearly strain hardening, the relation between the maximum shearing stress and maximum shearing strain is of the form illustrated graphically in Fig. 4-5.

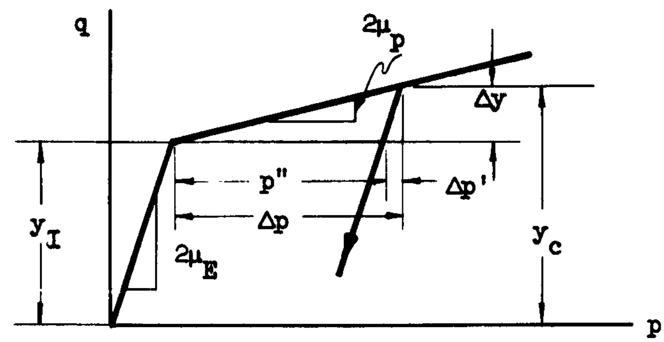

Fig. 4-5 Relation Between Maximum Shearing Stress and Strain in a Linearly Strain Hardening Medium

We write

$$y_c = y_I + \Delta y \quad , \qquad (a)$$

where y_c is the current yield stress, y_I is the initial yield stress which is equal to $y_o - BT$, and Δy is the increase in yield stress due to strain hardening. Similarly, we let

$$\Delta p = p'' + \Delta p' \quad , \qquad (b)$$

where Δp is the increase in total strain corresponding to the increase in yield stress, p'' is the plastic component of the maximum shearing strain, and $\Delta p'$ is the corresponding increase in the elastic component of the maximum shearing strain.

Referring to Fig. 4-5, we note that

$$\Delta y = 2 \mu_E \Delta p' \quad , \tag{c}$$

and

$$\Delta y = 2 \mu_p (p'' + \Delta p') \tag{d}$$

We eliminate $\Delta p'$ from (d) by combining (c) and (d),

$$\Delta y = \frac{1}{1-s} 2 \mu_p p'' \quad , \tag{e}$$

where

$$s = \frac{\mu_p}{\mu_E}$$

From (a) and (e), the current yield stress is then obtained in the form

$$y_c = y_o - BT + \frac{1}{1-s} 2 \mu_p p'' \tag{f}$$

Since, by (4.3.13), the maximum elastic shearing strain p' is defined as $\varepsilon_{xx}/2$, we have that

$$\Delta p' = \Delta \varepsilon_{xx}/2 \tag{g}$$

Similarly, we write

$$p'' = r_{xx}/2 \tag{h}$$

where r_{xx} is the plastic strain component in the x direction. It remains to express p'' in terms of the strain hardening parameter ξ defined by (3.1.2).

Taking the plastic incompressibility, and the symmetry about the x axis into account, we readily find

$$s_{yy} = s_{zz} = -\frac{1}{2} s_{xx} \quad ,$$

Moreover,

$$s_{xy} = s_{yz} = s_{xz} = 0 \quad ,$$

and, by (3.1.2),

$$\xi = \int_0^t [s_{xx}{}^2 + 2(-\tfrac{1}{2} s_{xx})^2]^{1/2} \, dt$$

$$= (\tfrac{3}{2})^{1/2} \; r_{xx} \qquad \text{(i)}$$

Making use of (h) and (i), we rewrite (f) as

$$y_c = y_o - BT + \frac{1}{1-s} \, 2 \mu_p \frac{\xi}{\sqrt{6}} \qquad \text{(j)}$$

If the medium has not undergone previous plastic deformation, the parameter ξ is equal to zero, (j) reduces to (4.4.1).

4-43-2 <u>The Stress and Strain Rates Corresponding to Positive Temperature Rate</u>

The yield criterion of Tresca will be used here not only to provide a means for locating the elastic-plastic interface, but also to serve as yield function from which plastic stress strain relations are derived. We observe from the elastic solution that the stresses τ_{yy} and τ_{zz} are negative corresponding to rising temperature, and positive for falling temperature. The corresponding maximum shearing stresses are, therefore, given by $-\tau_{yy}/2$, $-\tau_{zz}/2$ in the first case, and by $\tau_{yy}/2$, $\tau_{zz}/2$ in the second case.

We now combine (3.3.3), (3.3.4) with (j), and obtain a non-isothermal yield condition in the form

$$F^{(1)} = f^{(1)} - H = \frac{v_x - v_y}{2} - (y_o - BT + \frac{1}{1-s}\ 2\mu_p \frac{\xi}{\sqrt{6}}) \qquad (4.4.8)$$

$$F^{(2)} = f^{(2)} - H = \frac{v_x - v_z}{2} - (y_o - BT + \frac{1}{1-s}\ 2\mu_p \frac{\xi}{\sqrt{6}}) \qquad (4.4.9)$$

where

$$f^{(1)} = \frac{v_x - v_y}{2} = - \frac{\tau_{yy}}{2} \quad ,$$

$$f^{(2)} = \frac{v_x - v_z}{2} = - \frac{\tau_{zz}}{2} \quad , \qquad (4.4.10)$$

$$H = y_c = (y_o - BT + \frac{1}{1-s}\ 2\mu_p \frac{\xi}{\sqrt{6}})$$

In particular, since

$$\tau_{yy} = \tau_{zz} = \tau$$

we have that

$$f^{(1)} = f^{(2)} = f = - \tau/2 \quad , \quad F^{(1)} = F^{(2)} = F \qquad (4.4.11)$$

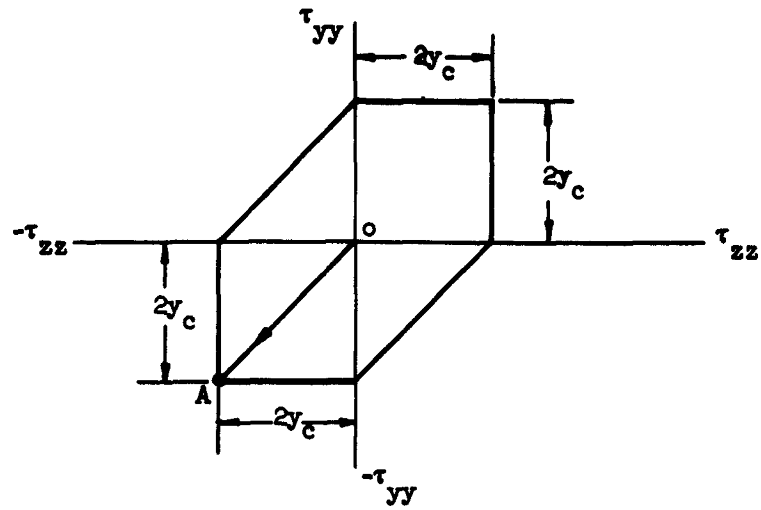

Fig. 4-6 Yield Surface and the Path of Loading for the Present Problem

From (4.4.11) follows that the state of stress in the plastic region corresponds to the yield corner which is represented by the point A in Fig. 4-6. The plastic stress strain relations may thus be obtained from (3.3.1), (3.3.2), and (3.2.8):

$$s_{ij} = \sum_{\gamma=1}^{2} s_{ij}^{(\gamma)}$$

$$s_{ij}^{(\gamma)} = \frac{\partial f^{(\gamma)}/\partial \tau_{ij}}{\partial H/\partial \xi \, (\partial f^{(\gamma)}/\partial \tau_{mn} \, \partial f^{(\gamma)}/\partial \tau_{mn})^{1/2}} (\partial F/\partial v_{kl} \, \dot{v}_{kl} + \partial F/\partial T \, \dot{T}) \tag{4.4.13}$$

where $f^{(\gamma)}$ are given by (4.4.8), (4.4.9) and (4.4.10). The flow rule (4.4.13) is complemented by Hooke's Law, (4.3.2), and the condition that the total strains in the y and z directions vanishes. These conditions furnish five equations for the five unknowns $\mathcal{E}_{xx}$, s_{xx}, $\mathcal{E}_{yy}$,

s_{yy} and τ_{yy} in the plastic region. Specifically, we find*

$$s_{xx} = \frac{1}{\frac{1}{1-s}\,\mu_p\left(\frac{1}{3}\right)^{1/2}}\left(-\frac{1}{2}\,\dot{\tau}_{yy} + B\dot{T}\right) \qquad \text{(a)}$$

$$s_{yy} = s_{zz} = -\frac{1}{2}\,\frac{1}{\frac{1}{1-s}\,\mu_p\left(\frac{1}{3}\right)^{1/2}}\left(-\frac{1}{2}\,\dot{\tau}_{yy} + B\dot{T}\right) \qquad \text{(b)}$$

$$(4.4.14)$$

$$\dot{\mathcal{E}}_{xx} = 2\,\nu_1/E_1\ \dot{\tau}_{tt} + \alpha\ \dot{T} \qquad \text{(c)}$$

$$\dot{\mathcal{E}}_{yy} = (1-\nu_1)/E_1\ \dot{\tau}_{yy} + \alpha\ \dot{T} \qquad \text{(d)}$$

$$s_{yy} = -\dot{\mathcal{E}}_{yy} \qquad \text{(e)}$$

Since the stress $\tau_{yy} = \tau_{zz}$ and the total strain ϵ_{xx} are of primary importance, we solve for them from (4.4.14), and obtain

$$\dot{\tau}_{yy} = \dot{\tau}_{zz} = M\,\dot{T} \quad , \qquad (4.4.15)$$

$$\dot{\epsilon}_{xx} = s_{xx} + \dot{\mathcal{E}}_{xx} = N\,\dot{T} \quad , \qquad (4.4.16)$$

---

* The derivation of (a) and (b) in (4.4.14) is given in Appendix B.

where

$$M = \frac{2\,BE_1\,(1-s) - 2\,E_1\,\alpha\,\mu_p\,(4/3)^{1/2}}{E_1\,(1-s) + 2\,(1-\nu_1)\,\mu_p\,(4/3)^{1/2}} \quad ,$$

$$N = \left[\alpha + \frac{B\,(1-s)\,(3)^{1/2}}{\mu_p}\right] - M\left[\frac{(1-s)\,(3)^{1/2}}{2\,\mu_p} + \frac{2\,\nu_1}{E_1}\right] \tag{4.4.17}$$

We emphasize that equations (4.4.15) and (4.4.16) are applicable only to the plastic regions where plastic flow is induced by rising temperature. Moreover, with reference to (4.4.15) and (4.4.17), we note that a positive temperature rate may give rise to a positive or negative stress rate depending on whether the effect of temperature upon yield stress dominates or the effect of strain hardening upon yield stress dominates. In thefirst case, the stress τ_{yy} in the plastic region will always be less than the value it assumed at initial yielding.

The stress and strain distribution in the plastic region may be obtained by integration of (4.4.15) and (4.4.16).

4-43-3 <u>The Stress and Strain Rates Corresponding to Negative Temperature Rate</u>

If the boundary of the half space is cooled, the stress in the y and z directions becomes tensile and the maximum shearing stress becomes $\tau_{yy}/2 = \tau_{zz}/2$ in the place of $-\tau_{yy}/2 = -\tau_{zz}/2$ found for the case of heating. The yield functions corresponding to (4.4.8) and (4.4.9) now assume the forms

$$F^{(1)'} = f^{(1)'} - H = \frac{v_y - v_x}{2} - (y_o - BT + \frac{1}{1-s}\ 2\mu_p\ \frac{\bar{\epsilon}}{\sqrt{6}}) \qquad (4.4.18)$$

$$F^{(2)'} = f^{(2)'} - H = \frac{v_z - v_x}{2} - (y_o - BT + \frac{1}{1-s}\ 2\ \mu_p\ \frac{\bar{\epsilon}}{\sqrt{6}}) \qquad (4.4.19)$$

where

$$f^{(1)'} = \frac{v_y - v_x}{2} = \frac{\tau_{yy}}{2}$$

$$f^{(2)'} = \frac{v_z - v_x}{2} = \frac{\tau_{zz}}{2}$$

$$H = y_c = y_o - BT + \frac{1}{1-s}\ 2\ \mu_p\ \frac{\bar{\epsilon}}{\sqrt{6}}$$

Again, since

$$\tau_{yy} = \tau_{zz}$$

we have that

$$f^{(1)'} = f^{(2)'} = f' \ , \quad F^{(1)'} = F^{(2)'} = F$$

Proceeding as before, it is found that the stress rate and strain rate are related to the temperature rate by

$$\dot{\tau}_{yy} = M'\dot{T} \qquad (4.4.20)$$

$$\dot{\epsilon}_{xx} = N'\dot{T} \qquad (4.4.21)$$

where

$$M' = \frac{-2\ B\ E_1(1-s)\ -\ 2E_1\ \alpha\ \mu_p\ (4/3)^{1/2}}{E_1\ (1-s) + 2\ \mu_p\ (1-\nu_1)\ (4/3)^{1/2}}$$

(4.4.22)

$$N' = [\ \alpha - \frac{B\ (1-s)\ (3)^{1/2}}{\mu_p}\ ] - M' [\ \frac{(1-s)\ (3)^{1/2}}{2\ \mu_p} + \frac{2\nu_1}{E_1}\ ]$$

Relations (4.4.20), (4.4.21), (4.4.22) may only be used to describe the stress and strain rates in any plastic regions where plastic flow is induced by falling temperature.

If the yield stress is independent of temperature, the response of the medium should be the same regardless of whether the external excitation is a heating or a cooling pulse. This, in fact, is the case if the quantity B in (4.4.15) and (4.4.20) is set equal to zero, in which case these equations become identical; the same is true for (4.4.16) and (4.4.21). Furthermore, if in addition to setting B equal to zero in (4.4.15), (4.4.20), (4.4.16) and (4.4.21), the plastic shear modulus μ_p is set equal to the elastic shear modulus μ_E, these equations reduce to their elastic counterparts represented by (4.3.8) and (4.3.9).

4-43-4 The Criteria of Loading and Unloading

As was noted previously, the medium will undergo some plastic deformation in the region $a \leqslant x \leqslant R_1$ if temperature exceeds the value T_1 given by (4.4.6). For a point in this region, two instants are of prime importance: the instant at which plastic flow begins, and the instant at which plastic flow ceases. The first instant can be obtained from (4.4.2), whereas the second instant remains to be determined.

The loading and unloading criteria discussed in Chapter 3 may help to shed some light on the duration of plastic flow. From the relations (3.3.2), (4.4.8), (4.4.9), (4.4.11), (4.4.12) follows that, for the present case, these criteria may be expressed by‡

$$-\frac{1}{2}\,\dot{\tau}_{yy} + B\dot{T} > 0 \qquad \text{for loading}$$
$$-\frac{1}{2}\,\dot{\tau}_{yy} + B\dot{T} < 0 \qquad \text{for unloading} \tag{4.4.23}$$

Substituting (4.4.15) into (4.4.23), the cirteria become

$$(B - \frac{1}{2}M)\;\dot{T} > 0 \qquad \text{for loading}$$
$$(B - \frac{1}{2}M)\;\dot{T} < 0 \qquad \text{for unloading} \tag{4.4.24}$$

where M is given by the first of (4.4.17). The quantity inside the parentheses of (4.4.24) is a constant, and is found to be positive definite upon closer examination. Therefore, the criteria for loading and unloading may be expressed as

$$\dot{T} > 0 \qquad \text{for loading}$$
$$\dot{T} < 0 \qquad \text{for unloading} \tag{4.4.25}$$

‡ The derivation is given in Appendix B.

It is thus seen from (4.4.25) that, at any location in the region represented by $a \leq x \leq R_1$, plastic flow begins at the instant t_{1x} at which the temperature T is at T_1, and terminates at the instant t_{mx} for which the temperature is at its maximum T_{mx}.

4-43-5 Solution for the Plastic Region

The stress τ_{yy} and the total strain ϵ_{xx} in the plastic region may be obtained by integrating (4.4.15) and (4.4.16) between the limits t_{1x} and t where t_{1x} is the instant of incipient yielding at x. By (4.3.8), (4.3.12) and (4.4.6), we obtain after appropriate integrations

$$\tau_{yy}(x,t) = M\,T(x,t) - \left[2\,y_o + \frac{y_o}{B + E_1\alpha/2(1-\nu_1)}(M - 2B)\right] \qquad (4.4.26)$$

$$\epsilon_{xx}(x,t) = N\,T(x,t) - \left[\left(N - \frac{1+\nu_1}{1-\nu_1}\alpha\right)\frac{y_o}{B + E_1\,\alpha/2(1-\nu_1)}\right] \qquad (4.4.27)$$

where M and N are given by (4.4.17).

The displacement U_{xx}, denoted by U, is obtained in the plastic region by integrating (4.4.27) with respect to x between the limits ρ_1 and x

$$U(x,t) = \left(N - \frac{1+\nu_1}{1-\nu_1}\right)\frac{y_o}{B + E_1\,\alpha/2(1-\nu_1)}(\rho_1 - x)$$

$$- N\,a\,T_m\left(\frac{\pi}{2}e\right)^{1/2}\left[\psi\left(\frac{\rho_1}{2\sqrt{\kappa t}}\right) - \psi\left(\frac{x}{2\sqrt{\kappa t}}\right)\right]$$

$$+ g(\rho_1,t)\,, \qquad (4.4.28)$$

where $\psi(s)$ is the error function defined by (4.3.15). The function $g(\rho_1,t)$ may be determined by considering the condition of continuity of displacements at the elastic plastic interface $x = \rho_1$, namely

$$U_{elastic} = U_{plastic} \quad \text{at} \quad x = \rho_1 \tag{4.4.29}$$

From (4.4.28), (4.4.29) and (4.3.15) then follows the result

$$g(\rho_1,t) = \frac{1+\nu_1}{1-\nu_1}\, a\, \alpha\, T_m \left(\frac{\pi}{2}e\right)^{1/2} \left[\psi\left(\frac{\rho_1}{2\sqrt{\kappa t}}\right) - 1\right] \tag{4.4.30}$$

The complete solution for the time interval $t_1 \leq t \leq t_m$ is given by (4.3.8), (4.3.9), (4.3.15) for the elastic region and by (4.3.26), (4.4.27), (4.4.28), (4.4.29) for the plastic region.

4-5 The Elastic-Plastic-Elastic Regime

4-51 The Onset of Elastic Unloading

This section is concerned with the response of the material at time $t > t_m$ where t_m is the instant at which the temperature at the boundary is at its maximum. We recall here the criteria (4.4.25) for loading and unloading, and observe that any point in the region $a \leq x \leq R_1$ unloading is imminent when $\dot{T}$ becomes zero. In other words unloading impends as soon as the temperature at any location in the region $a \leq x \leq R_1$ has attained its peak value.

From inspection of (4.2.3) and (4.2.4) it is found that the temperature first attains its maximum at the boundary at the instant t_m given by

$$t_m = \frac{a^2}{2\kappa} \tag{4.2.3)R}$$

At this very instant the temperature at the boundary begins to decrease, and an elastic unloading front starts to progress into the medium. The unloading front eventually overtakes the front of plastic loading at the instant

$$t_{mR_1} = \frac{R_1^{\,2}}{2K} \qquad (4.2.4)R$$

and at the position $x = R_1$ of the half space.

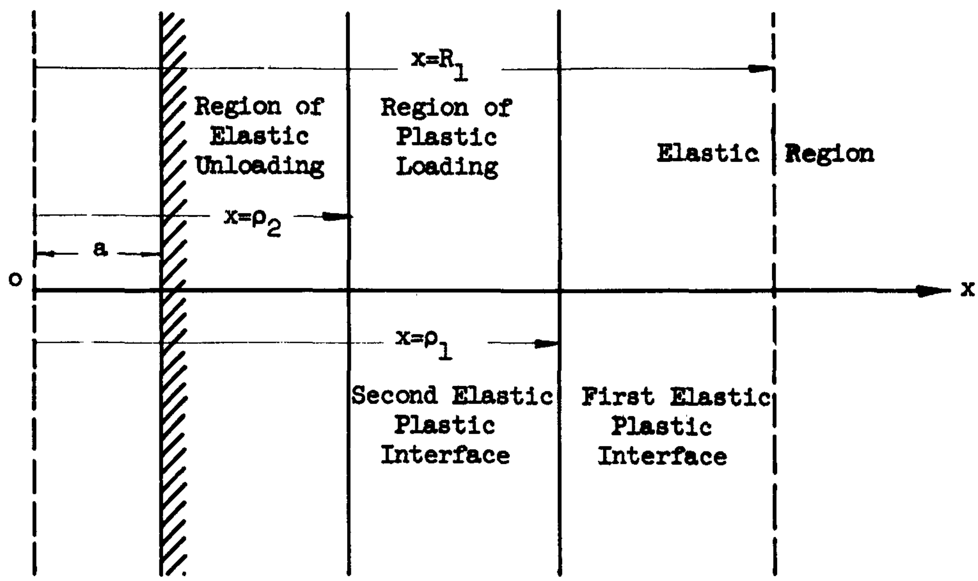

Fig. 4-7 Regions of Loading and Unloading in the Elastic-Plastic-Elastic Regime.

4-52 <u>The Elastic Region</u>

For time $t > t_m$, the elastic region still lies beyond the elastic plastic interface the position of which is denoted by ρ_1, and may be found from (4.4.5). The solution for this region is identical to that derived in section 4-3.

4-53 The Plastic Region of Loading

The plastic region now lies between the two interfaces ρ_1 and ρ_2, where ρ_2 denotes the position of the elastic unloading front. In this region the equations (4.4.26), (4.4.27) and (4.4.28), derived in the last section, still represent a valid solution.

Let us recall that the unloading front is located at a point where the temperature is at its maximum. For a given position x, the instant t_{mx} at which the temperature attains its maximum is found from (4.2.4) to be

$$t_{mx} = x^2/2\kappa \tag{4.2.4)R}$$

Therefore, the position $x = \rho_2$ of the unloading front is given by

$$\rho_2^{\,2} = 2\kappa\, t \quad , \tag{4.5.1}$$

$$t_{mR_1} \geq t \geq t_m \quad ,$$

where

$$t_{mR_1} = R_1^{\,2}/2\kappa \quad , \qquad t_m = a^2/2\kappa$$

The position $x = \rho_1$ of the plastic loading front can be determined as before from (4.4.5).

5-54 The Elastic Region of Unloading

For any instant t between t_{mR_1} and t_m, three regions exist simultaneously in the half-space, namely, the elastic region of unloading, the plastic region of loading, and the elastic region. However, for $t > t_{mR_1}$ only the elastic region of unloading and the elastic region remain. Replacing x by R_1 in (4.2.4), the instant t_{mR_1} may be obtained by substituting the expression for R_1 given by (4.4.7) in the resulting equation, thus,

$$t_{mR_1} = \frac{1}{2\kappa} \left\{ a \frac{T_m}{y_o} \left[B + \frac{E_1 \alpha}{2(1-\nu_1)} \right] \right\}^2 \qquad (4.5.2)$$

4-55 <u>The Transient Solution in the Elastic Region of Unloading</u>

The stress and strain rates in the elastic region of unloading may be obtained from (4.3.8) and (4.3.9) as

$$\dot{\tau}(x,t) = - \frac{E_1 \alpha}{1-\nu_1} \dot{T}(x,t) \qquad (4.5.3)$$

$$\dot{\epsilon}(x,t) = \frac{1+\nu_1}{1-\nu_1} \alpha \dot{T}(x,t) \qquad (4.5.4)$$

We shall take as the limits of integration t and t_{mx}, where t is greater than t_{mx}. We also recall that, by (4.2.6), the maximum temperature $T(x, t_{mx})$ is equal to $T_m a/x$. Then using for $\tau(x, t_{mx})$, $\epsilon(x, t_{mx})$ the results (4.4.26), (4.4.27), integration of (4.5.3), (4.5.4) yields

$$\tau(x,t) = - \frac{E_1 \alpha}{1-\nu_1} T(x,t) + \left\{ \left(\frac{\alpha E_1}{1-\nu_1} + M \right) T_m \frac{a}{x} - \left[2y_o + \frac{y_o}{B + E_1 \alpha / 2(1-\nu_1)} (M - 2B) \right] \right\} , \qquad (4.5.5)$$

for $t > t_{mx}$

$$\epsilon(x,t) = \frac{1+\mathcal{V}_1}{1-\mathcal{V}_1}\ \alpha\ T(x,t) + [(N - \frac{1+\mathcal{V}_1}{1-\mathcal{V}_1}\ \alpha)\ (T_m\ \frac{a}{x} - \frac{y_o}{B + E_1\alpha/2(1-\mathcal{V}_1)})]$$

$$\text{for}\quad t > t_{mx} \quad . \tag{4.5.6}$$

In order to obtain the displacement U, we integrate (4.5.6) between the limits ρ_i and x and obtain

$$U(x,t) = \frac{1+\mathcal{V}_1}{1-\mathcal{V}_1}\ a\ \alpha\ T_m\ (\frac{\pi}{2}\ e)^{1/2} [\ \psi(\frac{x}{2\sqrt{\kappa t}}) - \psi(\frac{\rho_i}{2\sqrt{\kappa t}})\ ]$$

$$+ \left\{ (N - \frac{1+\mathcal{V}_1}{1-\mathcal{V}_1}\ \alpha)\ [T_m\ a\ \ln\ \frac{x}{\rho_i} - \frac{y_o}{B + E_1\ \alpha/2(1-\mathcal{V}_1)}\ (x - \rho_i)] \right\}$$

$$+ \ g\ (\rho_i,\ t) \tag{4.5.7}$$

where ρ_i is defined as follows:

$$\rho_i = \begin{cases} \rho_2 & \text{for} \quad t_{mR_1} > t > t_n \\ R_1 & \text{for} \quad t \geqslant t_{mR_1} \end{cases}$$

Formally the function $g(\rho_i,\ t)$ in (4.5.7) is determined from the conditions of continuity of displacements across the interface $x = \rho_i$,

$$U_{\text{elastic unloading}} = U_{\text{plastic}} \quad ,$$

$$\text{at}\quad x = \rho_2 \quad \text{for}\quad t_{mR_1} > t > t_m \quad , \tag{4.5.8}$$

and

$$U_{\text{elastic unloading}} = U_{\text{elastic}} \quad ,$$

$$\text{at} \quad x = R_1 \qquad \text{for} \qquad t \geq t_{mR_1} \quad . \tag{4.5.9}$$

Thus, by (4.5.7), (4.5.8), (4.5.9), (4.4.28), (4.4.30) and (4.3.15),

$$g(\rho_2,t) = (N - \frac{1+\nu_1}{1-\nu_1}\ \alpha)\ \frac{y_o}{B + E_1\ \alpha/2(1-\nu_1)}\ (\rho_1 - \rho_2)$$

$$- N\ a\ T_m\ (\ \frac{\pi}{2}\ e)^{1/2}\ [\psi(\frac{\rho_1}{2\sqrt{\kappa t}}) - \psi(\frac{\rho_2}{2\sqrt{\kappa t}})]$$

$$+ \frac{1+\nu_1}{1-\nu_1}\ a\ \alpha\ T_n\ (\ \frac{\pi}{2}\ e)^{1/2}\ [\psi(\frac{\rho_1}{2\sqrt{\kappa t}}) - 1] \tag{4.5.10}$$

and

$$g(R_1,t) = \frac{1+\nu_1}{1-\nu_1}\ a\ \alpha\ T_m\ (\ \frac{\pi}{2}\ e)^{1/2}\ [\psi(\frac{R_1}{2\sqrt{\kappa t}}) - 1] \tag{4.5.11}$$

4-56 <u>Steady State Solution in the Elastic Region of Unloading</u>

Expressions for residual stress τ_R, strain ϵ_R and permanent deformation U_R may now be derived from (4.5.5), (4.5.6), (4.5.7) and (4.5.11) by letting the time t in these equations tend to infinity. We readily find that

$$\tau_R(x) = (M + \frac{\alpha E_1}{1-\nu_1})\ T_m\ \frac{a}{x}$$

$$- [2\ y_o + \frac{y_o}{B + E_1\alpha/2(1-\nu_1)}\ (M - 2B)] \quad , \tag{4.5.12}$$

$$\epsilon_R(x) = (N - \frac{1+\nu_1}{1-\nu_1}\alpha) \quad [T_m \frac{a}{x} - \frac{y_o}{B + E_1\ \alpha/2(1-\nu_1)}] \tag{4.5.13}$$

$$U_R(x) = (N - \frac{1+\nu_1}{1-\nu_1}\alpha) \quad [T_m\ a \ln \frac{x}{R_1} - \frac{y_o}{B + E_1\ \alpha/2(1-\nu_1)}(x-R_1)]$$

$$- \frac{1+\nu_1}{1-\nu_1}\ a\ \alpha\ T_m\ (\frac{\pi}{2}\ e)^{1/2} \tag{4.5.14}$$

where, by (4.4.7), the quantity R_1 is given by

$$R_1 = a\ \frac{T_m}{y_o}\ [B + \frac{E_1\ \alpha}{2(1-\nu_1)}] \tag{4.4.7)R}$$

We observe from (4.5.12), (4.5.13) and (4.5.14) that the residual stress, strain and deformation decrease from their maximum values at the boundary to zero at $x = R_1$. The last term in (4.5.14) represents the uniform expansion of the material, and is due to the assumed vanishing of displacement at infinity*.

With the transient solution given by (4.5.5), (4.5.6), (4.5.7), and the steady state solution given by (4.5.12), (4.5.13), (4.5.14), solution for the elastic region of unloading is completely determined. However, it should be noted here that the relations just mentioned above are applicable to any position in the region of unloading only if the material there is in the elastic state.

4-6 Extension of the Solution to Higher Boundary Temperature

In the preceding sections of this chapter, it was tacitly assumed that the maximum boundary temperature T_m is of such a magnitude that

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*cf. The equations (4.3.15) and (4.3.16).

no yielding occurs in the region of unloading. However, an inspection of the equation (4.5.12) for residual stresses shows that the residual stresses increase with increasing values of the maximum boundary temperature T_m. It is, therefore, possible that for sufficiently large T_m, the process of elastic unloading may be followed by plastic flow. In what follows this possibility will be fully investigated.

4-61 Yielding in the Region of Elastic Unloading

As was noted in section 4-4, unloading at the point x begins at the instant t_{mx} when the temperature there has reached its maximum value. Let us now define t_{px} to be the instant at which plastic flow occurs during unloading at x. If we account for the temperature dependence of yield stress, then, as may be seen from Fig. 4-8, yielding in the opposite sense will impend, if the stress τ_{yy} or τ_{zz} has changed by the amount

$$2/\tau_{yy}(x,t_{mx})/ + 2B[T(x,t_{mx}) - T(x,t_{px})]$$

Equivalently, at the instant of incipient yielding in opposite sense, the maximum shearing stress $q(x,t)$ has changed by the amount

$$q(x,t) = 2/\frac{\tau_{yy}(x,t_{mx})}{2}/ + B\ [T(x,t_{mx}) - T(x,t_{px})]$$

during the time interval from the start of unloading at t_{mx} until the instant t_{px}. The mathematical expression for the criterion of yielding in the region of elastic unloading is then as follows

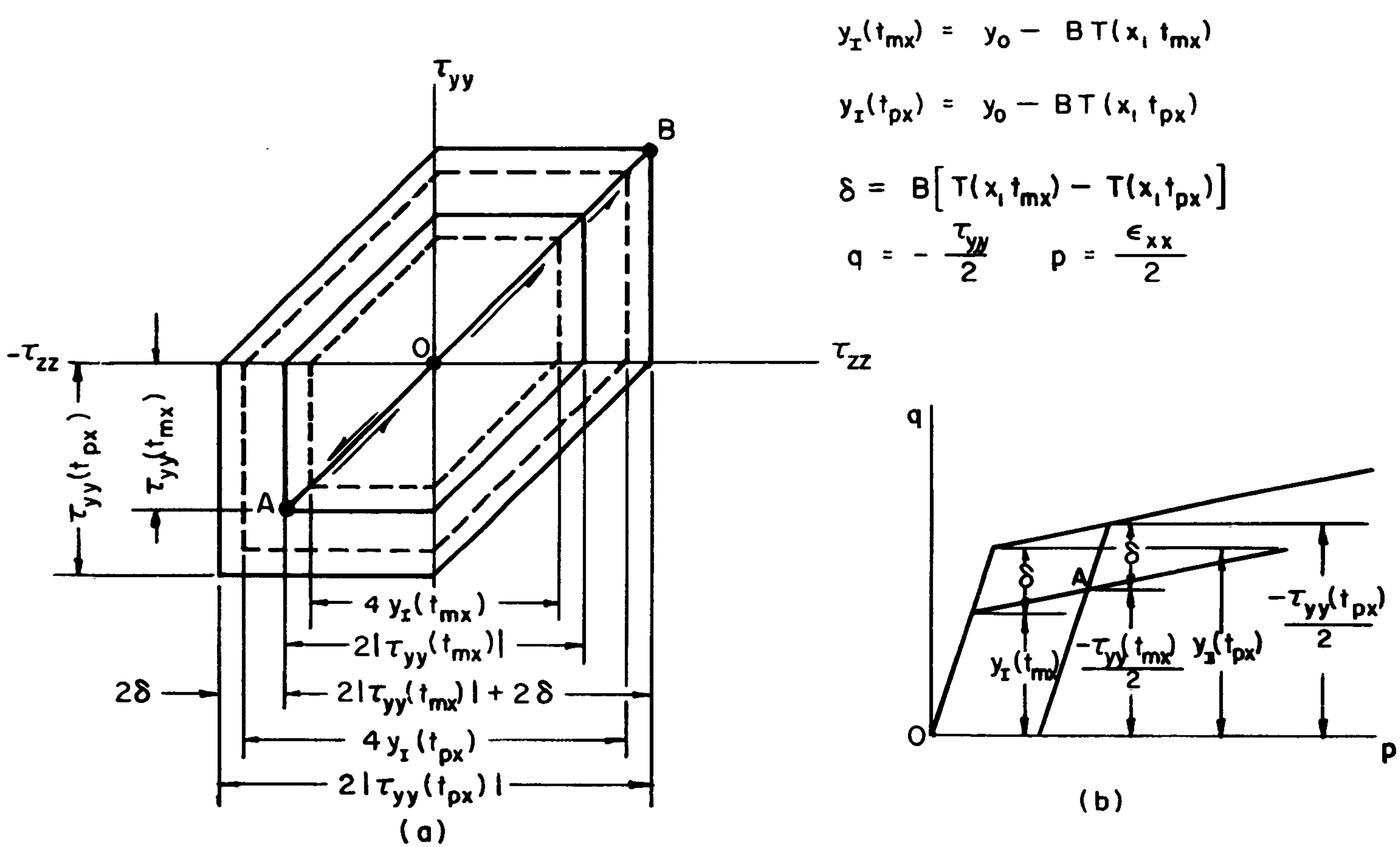

FIG. 4-8 PATH OF LOADING, UNLOADING OF A STRAIN HARDENING MATERIAL

$$\tau_{yy}(x,t_{px}) - \tau_{yy}(x,t_{mx}) =$$

$$2/\tau_{yy}(x,t_{mx})/ + 2B\,[T(x,t_{mx}) - T(x,t_{px})] \quad (4.6.1)$$

If equations (4.2.6), (4.4.26) and (4.5.5) are substituted into (4.6.1), we obtain

$$(2B - \frac{E_1\alpha}{1-\nu_1})\,T(x,t_{px}) = (2B - \frac{E_1\alpha}{1-\nu_1} - 2M)\,T_m\,\frac{a}{x}$$

$$+ 2\,[2\,y_o + \frac{y_o}{B + E_1\alpha/2(1-\nu_1)}\,(M - 2B)] \qquad (4.6.2)$$

It is of interest to determine the maximum boundary temperature T_M for which the elastic unloading will not be followed by yielding. Setting $T(x,t_{px})$ in (4.6.2) equal to zero, we find the result

$$T_M = \frac{1}{[M + \alpha E_1/2(1-\nu_1) - B]}\,[2y_o + \frac{y_o}{B + E_1\,\alpha/2(1-\nu_1)}\,(M - 2B)] \qquad (4.6.3)$$

If the maximum boundary temperature T_m is greater than T_M, plastic flow will occur again during unloading. Let t_{pa} denote the instant of incipient plastic flow at the boundary $x = a$. Then both t_{pa} and t_{px} may be determined from (4.2.5) and (4.6.2):

$$\frac{e^{-a^2/4 \ t_{pa}}}{\sqrt{2\kappa \, t_{pa}}} = \frac{1}{(2B - E_1\alpha/1-\nu_1)\, T_m \, a\sqrt{e}} \quad \left(2B - \frac{\alpha E_1}{1-\nu_1} - 2M\right) T_m$$

$$+ \, 2\left[2\, y_o + \frac{y_o}{B + E_1 \, \alpha/2(1-\nu_1)} \, (M - 2B)\right] \tag{4.6.2a}$$

$$\frac{e^{-x^2/4\kappa \ t_{px}}}{\sqrt{2\kappa \, t_{px}}} = \frac{1}{(2B - E_1 \, \alpha/1-\nu_1)\, T_m \, a\sqrt{e}} \quad \left(2B - \frac{\alpha E_1}{1-\nu_1} - 2M\right) T_m \, \frac{a}{x}$$

$$+ \, 2\left[2\, y_o + \frac{y_o}{B + E_1 \, \alpha/2(1-\nu_1)} \, (M - 2B)\right] \tag{4.6.2b}$$

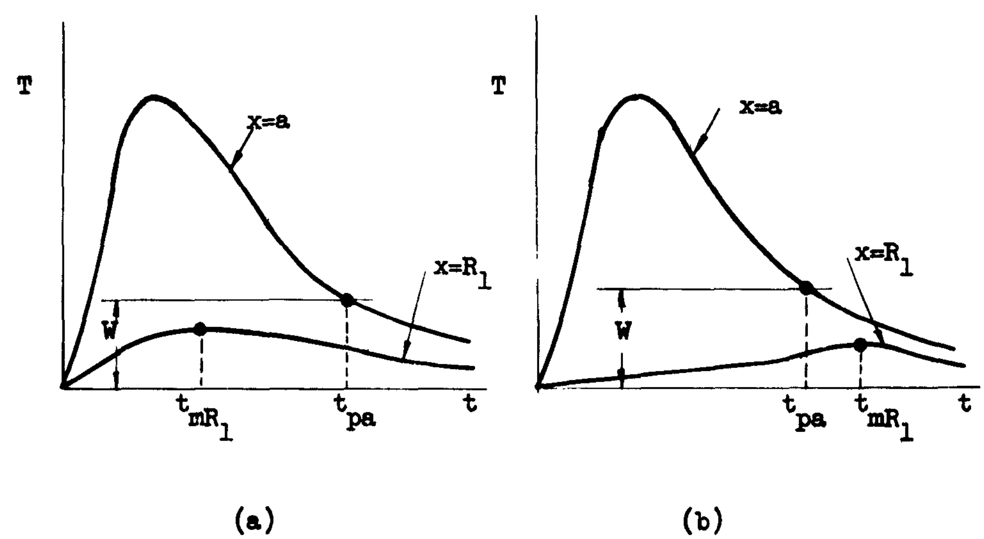

Fig. 4-9 Graphical Determination of t_{pa} in T-t Plots

Since a closed form solution for t_{pa} does not exist, we may determine t_{pa} graphically by drawing a horizontal line of height W, and intercepting the temperature curve for the boundary $x = a$ as shown in Fig. 4-9. Here, by (4.6.2),

$$W = T(a,t_{pa}) = (2B - \frac{E_1\alpha}{1-\nu_1} - 2M)\ T_m/(2B - \frac{\alpha E_1}{1-\nu_1})$$

$$+ 2\ [2y_o + \frac{y_o}{B + E_1\alpha/2(1-\nu_1)}\ (M - 2B)]/(2B - \frac{\alpha E_1}{1-\nu_1})$$

For $t > t_{pa}$, a new plastic interface, specified by $x = \rho_3$, propagates from the boundary into the interior of the half-space. The position ρ_3 can be found from (4.6.2b) if we there replace x by ρ_3, and t_{px} by t. Denoting the steady state value of ρ_3 by R_{11} and setting $T(x,t_{pa})$ equal to zero in (4.6.2), we obtain after rearranging

$$R_{11} = \frac{a\ T_m}{y_o}\ [B + \frac{E_1\alpha}{2(1-\nu_1)}]\ [1 - \frac{B + E_1\ \alpha/2(1-\nu_1)}{M + E_1\ \alpha/1-\nu_1}] \qquad (4.6.4)$$

We recall that the steady state position R_1 of the plastically deformed region is given by

$$R_1 = \frac{a\ T_m}{y_o}\ [B + \frac{E_1\ \alpha}{2(1-\nu_1)}] \qquad (4.4.7)R$$

From an inspection of (4.4.7) and (4.6.4), we conclude that the plastic region formed during unloading never extends to the boundary R_1 of the total plastically deformed region.

The total number of regions that may possibly exist in the half-space as a result of raising the maximum boundary temperature above the critical temperature T_M may be inferred from Fig. 4-9. The case when t_{pa} is greater than t_{mR_1} is illustrated in Fig. 4-9a. As may be readily seen, in the interval between t_{mR_1} and t_{pa} only two regions exist in the half-space; the elastic region of unloading and the elastic region. For $t > t_{pa}$, there exist three regions; the region of plastic loading in tension, the elastic region of unloading, and the elastic region. In Fig. 4-9b we have illustrated the case when t_{mR_1} is greater than t_{pa}. In the interval between the instants t_{pa} and t_{mR_1}, four regions exist simultaneously in the half-space; the region of plastic loading in tension, the region of elastic unloading, the region of plastic loading in compression, and the elastic region. However, for $t > t_{mR_1}$ only three regions remain, namely, the region of plastic loading in tension, the region of elastic unloading, and the elastic region.

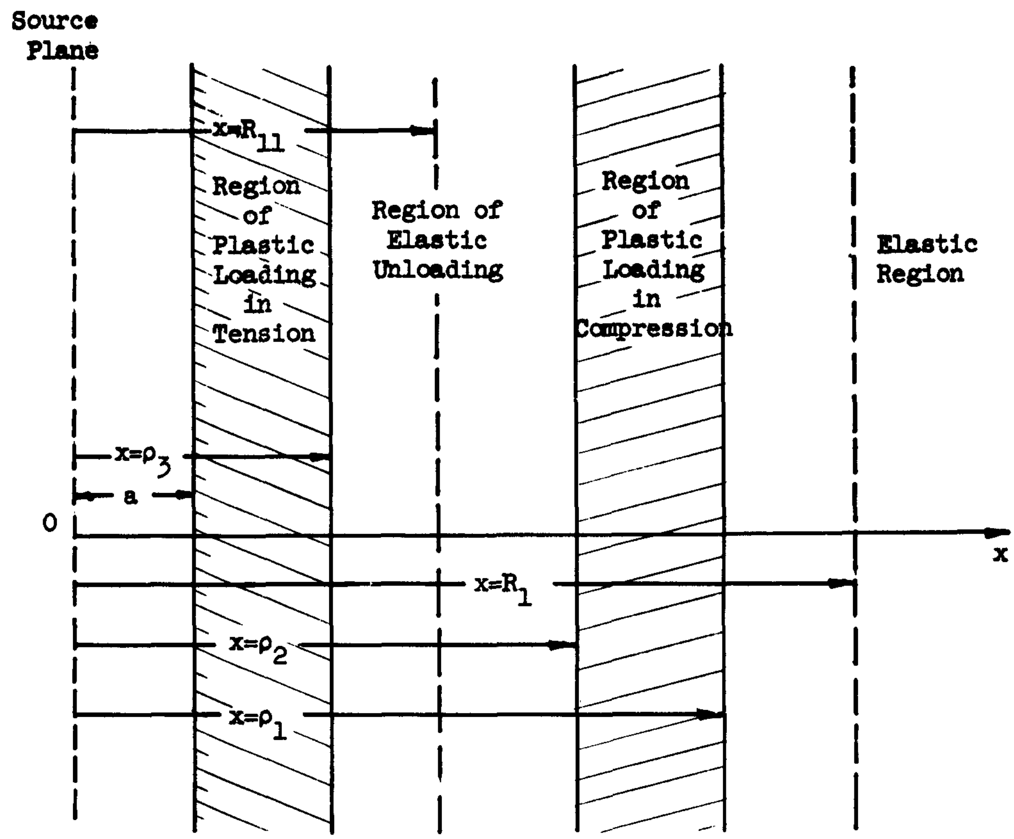

Fig. 4-10 Regions of Loading and Unloading in the Plastic-Elastic-Plastic-Elastic Regime

4-62 The Transient Solution in the Region of Plastic Loading in Tension

We note that, at the instant t_{px} of incipient yielding in the region of elastic unloading, the sign of the maximum shearing stress is opposite to what it was when yielding first occurred at that point, and at the instant t_{1x}. In the present case, the stresses in the y and z directions will be tensile rather than compressive; yielding is caused by the continuous decrease of temperature from its maximum value T_{mx}.

In order to find the stress and strain in this region, the equations (4.4.20) and (4.4.21) should be used in the place of (4.4.15) and (4.4.16). Recalling

$$\dot{\tau}_y = M' \dot{T} \quad , \tag{4.4.20)R}$$

$$\dot{\epsilon}_y = N' \dot{T} \quad , \tag{4.4.21)R}$$

where

$$M' = \frac{-2B\,E_1\,(1-s) - 2E_1\,\alpha\,\mu_p\,(4/3)^{1/2}}{E_1\,(1-s) + 2\,\mu_p\,(4/3)^{1/2}\,(1-\nu_1)} \,, \tag{4.4.22)R}$$

$$N' = \left[-\frac{(1-s)\,(3)^{1/2}}{2\,\mu_p} - \frac{2\,\nu_1}{E_1}\right] M' + \left[\frac{-B\,(1-s)\,(3)^{1/2}}{\mu_p} + \alpha\right]$$

the stress and strain in this region are determined by integrating (4.4.20) and (4.4.21) between the limits t and t_{px}. Thus, by (4.6.2), (4.5.5) and (4.5.6), it follows that

$$\tau_y(x,t) = M'T(x,t) + \left[\left(\frac{\alpha E_1}{1-\nu_1} + M\right) - \frac{M' + \alpha E_1/1-\nu_1}{2B - \alpha E_1/1-\nu_1}\left(2B - \frac{\alpha E_1}{1-\nu_1} - 2M\right)\right]$$
$$\cdot \left(T_m \frac{a}{x}\right) - \left[2\,\frac{M' + \alpha E_1/1-\nu_1}{2B - \alpha E_1/1-\nu_1} + 1\right] \cdot \left[2\,y_o\right.$$
$$\left. + \frac{y_o}{B + \alpha E_1/2(1-\nu_1)}\,(M - 2B)\right] \tag{4.6.5}$$

$$\epsilon(x,t) = N'T(x,t) + [(N - \frac{1+\nu_1}{1-\nu_1}\ \alpha) - \frac{N' - (1+\nu_1/1-\nu_1)\ \alpha}{2B - \alpha E_1/1-\nu_1}$$

$$\cdot(2B - \frac{\alpha E_1}{1-\nu_1} - 2M)]\ T_m\ \frac{a}{x} - \left\{ \frac{N' - (1+\nu_1/1-\nu_1)\ \alpha}{2B - \alpha E_1/1-\nu_1} \right.$$

$$\cdot[4\ y_o + \frac{2\ y_o}{B + \alpha E_1/2(1-\nu_1)}\ (M - 2B)]$$

$$\left. + \frac{N - (1+\nu_1/1-\nu_1)\ \alpha}{B + \alpha E_1/2(1-\nu_1)}\ y_o \right\} \qquad (4.6.6)$$

The displacement U is obtained by integrating (4.6.6) between the limits x and ρ_3

$$U(x,t) = a\ N'T_m\ (\frac{\pi}{2}\ e)^{1/2}\ [\psi(\frac{x}{2\sqrt{\kappa t}}) - \psi(\frac{\rho_3}{2\sqrt{\kappa t}})]$$

$$+ [\frac{N' - (1+\nu_1/1-\nu_1)\alpha}{2B - \alpha E_1/1-\nu_1}\ (2B - \frac{\alpha E_1}{1-\nu_1} - 2M)$$

$$+ (N - \frac{1+\nu_1}{1-\nu_1}\ \alpha)]\ T_m\ a\ \ln\ \frac{x}{\rho_3} - \left\{ \frac{N' - (1+\nu_1/1-\nu_1)\ \alpha}{2B - \alpha E_1/1-\nu_1} \right.$$

$$\cdot[4\ y_o + \frac{2\ y_o}{B + \alpha E_1/2(1-\nu_1)}\ (M - 2B)]$$

$$\left. + \frac{N - (1+\nu_1/1-\nu_1)\ \alpha}{B + \alpha E_1/2(1-\nu_1)}\ y_o \right\} (x - \rho_3) + g(\rho_3,t) \qquad (4.67)$$

Here $g(\rho_3,t)$ is a function of time, and is determined by the condition of continuity of displacement across the interface:

$$U_{\substack{\text{plastic loading}\\ \text{in tension}}} = U_{\text{elastic loading}} \quad \text{at} \quad x = \rho_3 \tag{4.6.8}$$

Thus, by (4.5.7), (4.5.10), (4.6.7) and (4.6.8),

$$g(\rho_3, t) = \frac{1+\nu_1}{1-\nu_1} \; a \, \alpha \, T_m \left(\frac{\pi}{2} e \right)^{1/2} \left[\psi\left(\frac{\rho_3}{2\sqrt{\kappa t}}\right) - \psi\left(\frac{\rho_2}{2\sqrt{\kappa t}}\right) \right.$$

$$+ \psi\left(\frac{\rho_1}{2\sqrt{\kappa t}}\right) - 1 \Big] + \left(N - \frac{1+\nu_1}{1-\nu_1} \alpha\right) \left[\frac{y_o}{B + \alpha E_1/2(1-\nu_1)} (\rho_1 - \rho_3) \right.$$

$$\left. + a \, T_m \ln \frac{\rho_3}{\rho_2} \right] - a \, N \, T_m \left(\frac{\pi}{2} e \right)^{1/2} \left[\psi\left(\frac{\rho_1}{2\sqrt{\kappa t}}\right) \right.$$

$$\left. - \psi\left(\frac{\rho_2}{2\sqrt{\kappa t}}\right) \right] \tag{4.6.9}$$

for $t_{mR_1} > t > t_{pa}$

For the instants $t > t_{mR_1}$, the function $g(\rho_3, t)$ is the same as that given by (4.6.9) except that $\rho_1 = \rho_2 = R_1$. Specifically,

$$g(\rho_3, t) = \frac{1+\nu_1}{1-\nu_1} \; a \, \alpha \, T_m \left(\frac{\pi}{2} e \right)^{1/2} \left[\psi\left(\frac{\rho_3}{2\sqrt{\kappa t}}\right) - 1 \right] + \left(N - \frac{1+\nu_1}{1-\nu_1} \alpha\right)$$

$$\cdot \left[a \, T_m \ln \frac{\rho_3}{R_1} - \frac{y_o}{B + \alpha E_1/2(1-\nu_1)} (\rho_3 - R_1) \right] \tag{4.6.10}$$

for $t > t_{pa} > t_{mR_1}$

The stress, strain and displacement for the region of plastic loading in tension for the cases $t_{pa} > t > t_{mR_1}$, $t_{mR_1} > t > t_{pa}$ and $t > t_{pa} > t_{mR_1}$ are now completely described by (4.6.5), (4.6.6), (4.6.7), (4.6.9) and (4.6.10).

4-63 The Steady State Solution in the Region of Plastic Loading in Tension

The residual stress τ_R and ϵ_R are obtained by setting the temperature $T(x,t)$ in (4.6.5) and (4.6.6) equal to zero, whereas the permanent deformation U_R is determined by letting the time t in (4.6.7) and (4.6.10) tend to infinity, and by replacing ρ_1 by R_1, ρ_3 with R_{11}. The results are

$$\tau_R(x) = [(M + \frac{\alpha E_1}{1-\nu_1}) - \frac{M' + \alpha E_1/1-\nu_1}{2B - \alpha E_1/1-\nu_1}(2B - \frac{\alpha E_1}{1-\nu_1} - 2M)]\, T_m \frac{a}{x}$$

$$- (2\frac{M' + \alpha E_1/1-\nu_1}{2B - \alpha E_1/1-\nu_1} + 1)\, [2\, y_o + \frac{y_o}{B + \alpha E_1/2(1-\nu_1)}(M - 2B)]$$

$$(4.6.11)$$

$$\epsilon_r(x) = [(N - \frac{1+\nu_1}{1-\nu_1}\alpha) - \frac{N' - (1+\nu_1/1-\nu_1)\,\alpha}{2B - \alpha E_1/1-\nu_1}(2B - \frac{\alpha E_1}{1-\nu_1} - 2M)]\, T_m \frac{a}{x}$$

$$-\left\{\frac{N' - (1+\nu_1/1-\nu_1)\,\alpha}{2B - \alpha E_1/1-\nu_1}[4\, y_o + \frac{2\, y_o}{B + \alpha E_1/2(1-\nu_1)}(M - 2B)]\right.$$

$$\left. + \frac{N - (1+\nu_1/1-\nu_1)\,\alpha}{B + \alpha E_1/2(1-\nu_1)}\, y_o\right\} \qquad (4.6.12)$$

$$U_R(x) = [(N - \frac{1+\nu_1}{1-\nu_1}\ \alpha) - \frac{N' - (1+\nu_1/1-\nu_1)\ \alpha}{2B - \alpha E_1/1-\nu_1}\ (2B - \frac{\alpha E_1}{1-\nu_1} - 2M)]T_m a\ \ln \frac{x}{R_{11}}$$

$$+ \left\{ \frac{N' - (1+\nu_1/1-\nu_1)\ \alpha}{2B - \alpha E_1/1-\nu_1}\ [4\ y_o + \frac{2\ y_o}{B + \alpha E_1/2(1-\nu_1)}\ (M - 2B)] \right.$$

$$\left. + \frac{N - (1+\nu_1/1-\nu_1)\ \alpha}{B + \alpha E_1/2(1-\nu_1)}\ y_o \right\} (R_{11} - x) + [(N - \frac{1+\nu_1}{1-\nu_1}\alpha)\ a\ T_m\ \ln \frac{R_{11}}{R_1}$$

$$+ \frac{y_o}{B + \alpha E_1/\beta(1-\nu_1)}\ (R_1 - R_{11})] - \frac{1+\nu_1}{1-\nu_1}\ a\ \alpha\ T_m\ (\frac{\pi}{2}\ e)^{1/2}$$

(4.6.13)

The stresses, strains and displacements in the regions lying beyond ρ_3 are derivable from the relevant formulas in sections 4-3, 4-4 and 4-5.

Chapter 5. An Elastic Perfectly Plastic Response of a Half-Space to a Uniformly Applied Heat Pulse

5-1 Introduction

The solution for the response of a half-space subjected to a uniformly distributed heat pulse was presented in the preceding chapter; the medium was assumed there to be elastic, linearly strain hardening, and possessing a temperature dependent yield stress. In this chapter the same problem is reconsidered for the cases of perfectly plastic response with temperature dependent yield stress, and also assuming the yield stress to be constant.

The requisite solutions may be obtained by a direct calculation, or as special cases of the solution for strain hardening materials. Since the pattern of growth and decay of various elastic and plastic regions is essentially the same, we shall not give a detailed discussion of the solution, and limit the presentation to a statement of the solutions for stress, strain and displacement in the different regions.

The chapter is divided into two parts: In Part 1 we investigate an elastic perfectly plastic medium with a yield stress varying linearly with temperature, whereas in Part 2 the yield stress is taken to be constant. The elastic solution is the same for both cases, and is supplied by (4.3.8), (4.3.9) and (4.3.15).

5-2 Part 1. Reponse of an Elastic Perfectly Plastic Medium with a Yield Stress Varying Linearly with Temperature

5-21 The Plastic Region

As before, we use the yield criterion of Tresca, so that yielding is assumed to occur when the maximum shearing stress becomes equal to the current yield stress y_c in shear. The value of the yield stress y_c is now given by,

$$y_c = y_o - BT(x,t) \tag{5.21}$$

and, by (3.3.3),(3.3.4) the yield functions assume the form,

$$F^{(1)} = -\frac{\tau_y}{2} - (y_o - BT)$$

$$F^{(2)} = -\frac{\tau_z}{2} - (y_o - BT)$$

where (5.22)

$$f^{(1)} = -\frac{\tau_y}{2} = \frac{v_x - v_y}{2}$$

$$f^{(2)} = -\frac{\tau_z}{2} = \frac{v_x - v_z}{2}$$

Moreover, since $\tau_y = \tau_z$, we have that

$$f^{(1)} = f^{(2)} \quad , \quad F^{(1)} = F^{(2)} \tag{5.23}$$

In the present case the maximum shearing stress in the plastic region is always equal to the yield stress; therefore, by (4.3.12),

$$\tau_y(x,t) = -2\,[y_o - BT(x,t)] \tag{5.2.4}$$

The strains are obtained by (3.3.5), Hooke's Law, and the condition that the total strains in the y and z directions are zero. Specifically, from (5.2.2) follow the relations

$$s_x = \sum_{\alpha=1}^{2} \lambda^{(\alpha)} \frac{\partial f^{(\alpha)}}{\partial \tau_x} = \frac{1}{2}\lambda^{(1)} + \frac{1}{2}\lambda^{(2)} = \frac{1}{2}(\lambda+\lambda) = \lambda \qquad \text{(a)}$$

$$s_y = s_z = \sum_{\alpha=1}^{2} \lambda^{(\alpha)} \frac{\partial f^{(\alpha)}}{\partial \tau_y} = -\frac{1}{2}\lambda^{(1)} = -\frac{1}{2}\lambda^{(2)} = -\frac{1}{2}\lambda \qquad \text{(b)}$$

$$\dot{\varepsilon}_x = -\frac{2\nu_1}{E}\dot{\tau}_y + \alpha\dot{T} \qquad \text{(c)}$$

$$\dot{\varepsilon}_y = \frac{1-\nu_1}{E}\dot{\tau}_y + \alpha\dot{T} \qquad \text{(d)}$$

$$s_y = -\dot{\varepsilon}_y \qquad \text{(e)}$$

Moreover, (5.2.4) implies that

$$\dot{\tau}_y = 2B\dot{T} \qquad \text{(f)}$$

The solution of the five equations (a), (b), (c), (d) and (e) for the five unknowns s_x, s_y, ε_x, ε_y and λ then yields

$$\lambda = 2\dot{\tau}_y \frac{1-\nu_1}{E_1} + 2\alpha\dot{T} \quad , \qquad (5.2.5)$$

$$s_x = \lambda = \left[4B\frac{1-\nu_1}{E_1} + 2\alpha\right]\dot{T} \quad , \qquad \text{(g)}$$

$$s_y = -\frac{1}{2}\left[4B\frac{1-\nu_1}{E_1} + 2\alpha\right]\dot{T} \quad , \qquad \text{(h)}$$

$$\dot{\varepsilon}_x = \left(- \frac{4B\nu_1}{E_1} + \alpha \right) \dot{T} \quad , \tag{i}$$

$$\dot{\varepsilon}_y = - s_y = \left[2B \frac{1 - \nu_1}{E_1} + \alpha \right] \dot{T} \quad , \tag{j}$$

and the total strain rates are expressed by

$$\dot{\epsilon}_x = s_x + \dot{\varepsilon}x = \left[\frac{4(1 - 2\nu_1)}{E_1} B + 3\alpha \right] \dot{T} \tag{5.2.6}$$

$$\dot{\epsilon}_y = 0 \tag{5.2.7}$$

The corresponding relations for regions in which plastic flow is induced by cooling can be found without difficulty, keeping in mind, however, that during the decrease in temperature stresses in the y and z directions are tensile rather than compressive. In the place of (5.2.2) the yield functions are now given by

$$\begin{aligned} F^{(1)} &= \frac{\tau_y}{2} - (y_o - BT) \\ F^{(2)} &= \frac{\tau_z}{2} - (y_o - BT) \\ f^{(1)} &= \frac{\tau_y}{2} = \frac{v_y - v_x}{2} \\ f^{(2)} &= \frac{\tau_z}{2} = \frac{v_z - v_x}{2} \end{aligned} \tag{5.2.8}$$

Since $\tau_y = \tau_z$

$$f^{(1)} = f^{(2)} \quad , \quad F^{(1)} = F^{(2)} \tag{5.2.9}$$

The stress and strain rates in the region in which plastic flow is induced by decrease in temperature may now be expressed by

$$\begin{aligned} \tau_y &= 2\,(y_o - BT) \\ \dot{\tau}_y &= -2\,B\dot{T} \end{aligned} \qquad (5.2.10)$$

$$\lambda = [4B\,\frac{1-\nu_1}{E_1} - 2\,\alpha]\,\dot{T} \qquad (5.2.11)$$

$$\dot{\epsilon}_x = s_x + \dot{\mathcal{E}}_x = [-\frac{4(1-\nu_1)}{E_1}\,B + 3\,\alpha]\,\dot{T} \qquad (5.2.12)$$

In order to calculate the strains from (5.2.6), we integrate between the limits t and t_{1x}. Noting that

$$\epsilon_x(x,t_{1x}) = \frac{1+\nu_1}{1-\nu_1}\,\alpha\,T\,(x,\,t_{1x})$$

is given by the elastic solution (4.3.8), and that by (4.4.6),

$$T(x,t_{1x}) = T_1 = \frac{y_o}{B + \alpha E_1/2(1-\nu_1)} \qquad (4.4.6)\text{R}$$

the total strain is then found to be

$$\epsilon_x(x,t) = [\frac{4(1-2\nu_1)}{E_1}\,B + 3\,\alpha]\,T(x,t)$$
$$+ [\frac{1+\nu_1}{1-\nu_1}\,\alpha - \frac{4(1-2\,\nu_1)}{E_1}\,B - 3\,\alpha]\,\frac{y_o}{B + \alpha E_1/2(1-\nu_1)}$$

where

$$T(x,t) = T_m \frac{a\sqrt{e}}{\sqrt{2\kappa t}} \; e^{-x^2/4\kappa t} \tag{5.2.13}$$

The corresponding displacement is obtained by integrating (5.2.13) between the limits x and ρ_1:

$$U(x,t) = \left[\frac{4(1-2\nu_1)}{E_1} B + 3\alpha\right] T_m \, a \left(\frac{\pi}{2} e\right)^{1/2} \left[\Psi\left(\frac{x}{2\sqrt{\kappa t}}\right) - \Psi\left(\frac{\rho_1}{2\sqrt{\kappa t}}\right)\right]$$

$$+ \left[\frac{1+\nu_1}{1-\nu_1}\,\alpha - \frac{4(1-2\nu_1)}{E_1} - 3\alpha\right] \frac{y_o}{B + \alpha E_1/2(1-\nu_1)} (x - \rho_1)$$

$$+ \frac{1+\nu_1}{1-\nu_1}\, a\,\alpha\, T_m \left(\frac{\pi}{2} e\right)^{1/2} \left[\Psi\left(\frac{\rho_1}{2\sqrt{\kappa t}}\right) - 1\right]$$

for $\rho_1 \geqslant x \geqslant a \quad , \quad t_m \geqslant t > t_1$

(5.2.14)

or $\rho_1 \geqslant x \geqslant \rho_2 \quad t_{mR_1} > t > t_m$

where ρ_1 and ρ_2 are determined from (4.4.5) and (4.5.1), respectively.

5-22 <u>The Elastic Region of Unloading</u>

The stress and strain rates in this region are given by

$$\dot{\tau}_y(x,t) = -\frac{\alpha E_1}{1-\nu_1} \dot{T}(x,t) \tag{4.5.3)R}$$

$$\dot{\epsilon}_x(x,t) = \frac{1+\nu_1}{1-\nu_1} \dot{T}(x,t) \tag{4.5.4)R}$$

Integrating (4.5.3) and (4.5.4) between limits t and t_{mx}, where $t > t_{mx}$, we find, by (4.2.6), (5.2.4) and (5.2.13), that

$$\tau_y(x,t) = -\frac{\alpha E_1}{1-\nu_1} T(x,t) + [(\frac{\alpha E_1}{1-\nu_1} + 2B) T_m \frac{a}{x} - 2 y_o] \tag{5.2.16}$$

$$\epsilon_x(x,t) = \frac{1+\nu_1}{1-\nu_1} \alpha [T(x,t) - T_m \frac{a}{x} + \frac{y_o}{B + \alpha E_1/2(1-\nu_1)}]$$

$$+ [\frac{4(1-2\nu_1)}{E_1} B + 3\alpha] [T_m \frac{a}{x} - \frac{y_o}{B + \alpha E_1/2(1-\nu_1)}] \tag{5.2.17}$$

Both (5.2.16) and (5.2.17) are applicable for values of x such that $\rho_2 \geqslant x \geqslant a$, $t_{pa} > t > t_m$ and $\rho_2 > x > \rho_3$ for $t > t_{pa}$, where t_{pa} and ρ_3 are determined from (4.6.2a) and (4.6.2b) by setting there μ_p equal to zero.

Integration of (5.2.17) now gives

$$U(x,t) = \frac{1+\nu_1}{1-\nu_1} a \alpha T_m (\frac{\pi}{2} e)^{1/2} [\psi(\frac{x}{2\sqrt{\kappa t}}) - \psi(\frac{\rho_2}{2\sqrt{\kappa t}}) + \psi(\frac{\rho_1}{2\sqrt{\kappa t}}) - 1]$$

$$+ [\frac{4(1-2\nu_1)}{E_1} B + 3\alpha - \frac{1+\nu_1}{1-\nu_1} \alpha] [T_m a \ln \frac{x}{\rho_2}$$

$$- \frac{y_o}{B + \alpha E_1/2(1-\nu_1)} (x-\rho_1)] + [\frac{4(1-2\nu_1)}{E_1} B + 3\alpha] T_m a (\frac{\pi}{2} e)^{1/2} [\psi(\frac{\rho_2}{2\sqrt{\kappa t}})$$

$$- \psi(\frac{\rho_1}{2\sqrt{\kappa t}})] \tag{4.2.18}$$

for $\rho_2 \geqslant x \geqslant a \quad , \quad t_{pa} > t > t_m$

$\rho_2 \geqslant x \geqslant \rho_3 \quad , \quad t_{mR_1} > t > t_{pa}$

or

$$U(x,t) = \frac{1+\nu_1}{1-\nu_1} a \alpha T_m \left(\frac{\pi}{2} e\right)^{1/2} [\psi(\frac{x}{2\sqrt{\kappa t}}) - 1]$$

$$+ [\frac{4(1-2\nu_1)}{E_1} B + 3\alpha - \frac{1+\nu_1}{1-\nu_1}\alpha][T_m a \ln \frac{x}{R_1} + \frac{y_o}{B + \alpha E_1/2(1-\nu_1)}$$

$$(R_1 - x_1)] \tag{5.2.19}$$

for $R_1 \geqslant x > a \quad , \quad t_{pa} > t \geqslant t_{mR_1}$

$R_1 > x > \rho_3 \quad , \quad t > t_{pa} > t_{mR_1}$

where t_m, ρ_1, R_1, ρ_2, t_{mR_1} in (4.2.18) and (4.2.19) are determined from equations (4.2.4), (4.4.5), (4.4.7), (4.5.1), (4.5.2) respectively.

The residual stress, strain and permanent deformation obtained from (5.2.16), (5.2.17) and (5.2.19) are expressed by

$$\tau_R(x) = (\frac{\alpha E_1}{1-\nu_1} + 2B)\, T_m \frac{a}{x} - 2y_o \tag{5.2.20}$$

$$\epsilon_R(x) = [\frac{4(1-2\nu_1)}{E_1} B + 3\alpha - \frac{1+\nu_1}{1-\nu_1}\alpha][T_m \frac{a}{x} - \frac{y_o}{B + \alpha E_1/2(1-\nu_1)})]$$

(5.2.21)

$$U_R(x) = [\frac{4(1-2\nu_1)}{E_1} B + 3\alpha - \frac{1+\nu_1}{1-\nu_1}\alpha]\ [T_m\, a \ln \frac{x}{R_1}$$

$$+ \frac{y_o}{B + \alpha E_1/2(1-\nu_1)} (R_1 - x)] - \frac{1+\nu_1}{1-\nu_1}\, a\, \alpha\, T_m\, (\frac{\pi}{2} e)^{1/2}$$

$$(5.2.22)$$

The equations (5.2.20), (5.2.21) and (5.2.22) are valid only in the region $R_1 \geqslant x \geqslant a$ for $T_m < T_M$, and in the region $R_1 \geqslant x \geqslant R_{11}$ for $T_m > T_M$, where T_M is determined from (4.6.3) by setting μ_p in it equal to zero.

5-23 Plastic Region Formed During Unloading

If the maximum boundary temperature T_m is greater than T_M, given by (4.6.3) for the case when μ_p is zero, plastic flow occurs again during unloading. This flow begins at the boundary for the instant t_{pa}, and, subsequently, a new elastic-plastic interface, specified by $x = \rho_3$, propagates from the boundary into the interior. This interface eventually reaches the position $x = R_{11}$ as the temperature T of the medium decreases to zero. We let t_{px} be the instant at which yielding occurs at the position x during unloading. The values T_M, t_{px}, t_{pa}, ρ_3 and R_{11} are given by the following equations as a result of setting the shear modulus μ_p equal to zero in (4.6.3), (4.6.2), (4.6.2a), (4.6.2b) and (4.6.4):

$$T_M = \frac{4\, y_o}{(\alpha E_1/1-\nu_1 + 2B)} \qquad (5.2.23)$$

$$\left(2B - \frac{\alpha E_1}{1-\nu_1}\right) T_m \frac{a\, e^{1/2}}{\sqrt{2\kappa\, t_{px}}}\, e^{-x^2/4\kappa\, t_{px}} + \left(\frac{\alpha E_1}{1-\nu_1} + 2B\right) T_m \frac{a}{x}$$

$$= 4\, y_o \qquad \text{for} \quad T_m > T_M \tag{5.2.24}$$

$$\left(2B - \frac{\alpha E_1}{1-\nu_1}\right) T_m \frac{a\, e^{1/2}}{\sqrt{2\kappa\, t_{pa}}}\, e^{-a^2/4\kappa\, t_{pa}} + \left(\frac{\alpha E_1}{1-\nu_1} + 2B\right) T_m$$

$$= 4\, y_o \qquad \text{for} \quad T_m > T_M \tag{5.2.25}$$

$$\left(2B - \frac{\alpha E_1}{1-\nu_1}\right) T_M \frac{a\, e^{1/2}}{\sqrt{2\kappa t}}\, e^{-\rho_3^2/4\kappa t} + \left(\frac{\alpha E_1}{1-\nu_1} + 2B\right) T_m \frac{a}{\rho_3}$$

$$= 4\, y_o \qquad \text{for} \quad T_m > T_M \tag{5.2.26}$$

$$R_{11} = \frac{a\, T_m\, (\alpha E_1/1-\nu_1 + 2B)}{4\, y_o} \tag{5.2.27}$$

In the plastic region formed during unloading the stresses in the y and z directions are tensile, and expressed by

$$\tau_y(x,t') = \tau_z(x,t) = 2\,[y_o - BT\,(x,t)] \quad , \tag{5.2.28}$$

whereas the strain rate is given by

$$\dot{\epsilon}_x\,(x,t) = [3\,\alpha - \frac{4(1-2\,\nu_1)}{E_1}\,B]\,\dot{T}\,(x,t)\quad . \tag{5.2.12)R$$

If the equation (5.2.12) is integrated between the limits t and t_{px} where $t > t_{px}$, then, by (5.2.24) and (5.2.17), the following result is obtained:

$$\epsilon_x(x,t) = [3\,\alpha - \frac{4(1-2\,\nu_1)}{E_1}\,B]\,T(x,t) + [\frac{4(1-2\,\nu_1)}{E_1}\,B + 3\,\alpha$$

$$- \frac{1+\nu_1}{1-\nu_1}\,\alpha][T_m\,\frac{a}{x} - \frac{y_o}{B + \alpha E_1/2(1-\nu_1)}] + [\frac{4(1-2\,\nu_1)}{E_1}\,B$$

$$- 3\alpha + \frac{1+\nu_1}{1-\nu_1}\,\alpha]\,\frac{1}{2B - \alpha E_1/1-\nu_1}\,[4\,y_o - T_m\,\frac{a}{x}\,(2B + \frac{\alpha E_1}{1-\nu_1})]$$

$$\text{for}\quad \rho_3 \geq x \geq a \quad\text{and}\quad t > t_{pa} \tag{5.2.29}$$

Integration of (5.2.29) further gives

$$U(x,t) = [3\alpha - \frac{4(1-2\nu_1)}{E_1}\,B]\,a\,T_m\,(\,\frac{\pi}{2}\,e)^{1/2}\,[\psi\,(\frac{x}{2\sqrt{\kappa t}}) - \psi\,(\frac{\rho_3}{2\sqrt{\kappa t}})]$$

$$+ [\frac{4(1-2\nu_1)}{E_1}\,B + 3\alpha - \frac{1+\nu_1}{1-\nu_1}\,\alpha]\,[T_m\,a\,\ln\frac{x}{\rho_3} - \frac{y_o}{B + \alpha E_1/2(1-\nu_1)}$$

$$(x-\rho_3)] + [\frac{4(1-2\,\nu_1)}{E_1}\,B - 3\alpha + \frac{1+\nu_1}{1-\nu_1}\,\alpha]\,[\frac{4\,y_o}{2B-\alpha E_1/1-\nu_1}\,(x-\rho_3)$$

$$- \frac{2B + \alpha E_1/1-\nu_1}{2B - \alpha E_1/1-\nu_1}\,T_m\,a\,\ln\,\frac{x}{\rho_3}\,] + g\,(\rho_3,t)\quad\text{for}\quad \rho_3 \geq x \geq a \tag{5.2.30}$$

where

$$g(\rho_3,t) = \frac{1+\nu_1}{1-\nu_1} a \alpha T_m \left(\frac{\pi}{2} e\right)^{1/2} [\psi(\frac{\rho_3}{2\sqrt{\kappa t}}) - \psi(\frac{\rho_2}{2\sqrt{\kappa t}})$$

$$+ \psi(\frac{\rho_1}{2\sqrt{\kappa t}}) - 1] + [\frac{4(1-2\nu_1)}{E_1} B + 3\alpha - \frac{1+\nu_1}{1-\nu_1} \alpha][T_m a \ln \frac{\rho_3}{\rho_2}$$

$$- \frac{y_o}{B + \alpha E_1/2(1-\nu_1)}(\rho_3 - \rho_1)] - [\frac{4(1-2\nu_1)}{E_1} B + 3\alpha] T_m a \left(\frac{\pi}{2} e\right)^{1/2}$$

$$\cdot [\psi(\frac{\rho_1}{2\sqrt{\kappa t}}) - \psi(\frac{\rho_2}{2\sqrt{\kappa t}})]$$

$$\text{for} \quad t_{mR_1} > t > t_{pa} \tag{5.2.31}$$

Alternately,

$$g(\rho_3,t) = \frac{1+\nu_1}{1-\nu_1} a \alpha T_m \left(\frac{\pi}{2} e\right)^{1/2} [\psi(\frac{\rho_3}{2\sqrt{\kappa t}}) - 1]$$

$$+ [\frac{4(1-2\nu_1)}{E_1} B + 3\alpha - \frac{1+\nu_1}{1-\nu_1} \alpha] [T_m a \ln \frac{\rho_3}{R_1}$$

$$+ \frac{y_o}{B + \alpha E_1/2(1-\nu_1)} (R_1 - \rho_3)]$$

$$\text{for} \quad t > t_{pa} > t_{mR_1} \tag{5.2.32}$$

where t_{mR_1}, ρ_2, ρ_3 and t_{pa} in (5.2.29), (5.2.30), (5.2.31), (5.2.32) are determined from (4.2.4), (4.5.1), (5.2.26), (5.2.25), respectively.

The residual stress, strain and permanent deformation for this region are found from (5.2.28), (5.2.29), (5.2.30) and (5.2.32). Thus we obtain the following results

$$\tau_R = 2\, y_o \tag{5.2.33}$$

$$\epsilon_R(x) = [\frac{4(1-2\nu_1)}{E_1} B + 3\alpha - \frac{1+\nu_1}{1-\nu_1}\alpha][T_m \frac{a}{x} - \frac{y_o}{B + \alpha E_1/2(1-\nu_1)}]$$

$$+ [\frac{4(1-2\nu_1)}{E_1} B - 3\alpha + \frac{1+\nu_1}{1-\nu_1}\alpha] \frac{1}{2B - E_1\alpha/1-\nu_1} [4\, y_o$$

$$- T_m \frac{a}{x} (2B + \frac{\alpha E_1}{1-\nu_1})] \tag{5.2.34}$$

$$U_R(x) = [\frac{4(1-2\nu_1)}{E_1} B + 3\alpha - \frac{1+\nu_1}{1-\nu_1}\alpha] [T_m\, a \ln \frac{x}{R_1}$$

$$+ \frac{y_o}{B + \alpha E_1/2(1-\nu_1)} (R_1 - x)] + [\frac{4(1-2\nu_1)}{E_1} B - 3\alpha$$

$$+ \frac{1+\nu_1}{1-\nu_1}\alpha] [\frac{2B + \alpha E_1/1-\nu_1}{2B - \alpha E_1/1-\nu_1} T_m\, a \ln \frac{R_{11}}{x}$$

$$- \frac{4\, y_o}{2B - \alpha E_1/1-\nu_1} (R_{11} - x)] \tag{5.2.35}$$

Equations (5.2.23), (5.2.34) and (5.2.35) are valid only for the region $R_{11} \geq x \geq a$ where R_{11} is determined by (5.2.28).

5-3 Part 2. Response of an Elastic Perfectly Plastic Medium with a Constant Yield Stress

If the infinite half-space is subjected to a uniformly applied heat pulse at its boundary, and if the medium is assumed to be elastic, perfectly plastic with a constant yield stress, the solution is readily obtained as a special case of the results described in Part 1 of this chapter. Specifically, the equations describing the stress, strain and displacement in various regions for the medium considered here may be deduced by setting the quantity B equal to zero in the corresponding equations derived in Part 1 of this chapter. The results of this simplification will be presented in the subsequent sections, the elastic solution being again given by (4.3.8), (4.3.9) and (4.3.15).

5-31 The Plastic Region

In this region, by (5.2.10), (5.2.13), (5.2.14), the stress, strain and displacement are expressed as follows:

$$\tau_y(x,t) = -2\,y_o \tag{5.3.1}$$

$$\epsilon_x(x,t) = 3\alpha\, T(x,t) - \left(3 - \frac{1+\nu_1}{1-\nu_1}\right)\frac{2(1-\nu_1)}{E_1}\, y_o \tag{5.3.2}$$

$$U(x,t) = a\,\alpha\, T_m\left(\frac{\pi}{2}e\right)^{1/2}\left\{3\left[\psi\left(\frac{x}{2\sqrt{\kappa t}}\right) - \psi\left(\frac{\rho_1}{2\sqrt{\kappa t}}\right)\right] + \frac{1+\nu_1}{1-\nu_1}\left[\psi\left(\frac{\rho_1}{2\sqrt{\kappa t}}\right) - 1\right]\right\} + \left(3 - \frac{1+\nu_1}{1-\nu_1}\right)\frac{2(1-\nu_1)}{E}\, y_o\,(\rho_1 - x)$$

$$\text{for} \quad \rho_1 \geqslant x > a \quad , \quad t_m \geqslant t \geqslant t_1 \quad ,$$
$$\rho_1 \geqslant x \geqslant \rho_2 \quad , \quad t_{mR_1} \geqslant t \geqslant t_m \quad . \tag{5.3.3}$$

where ρ_2 is determined from (4.5.1) and ρ_1 from (4.4.5) by setting B there equal to zero.

5-32 <u>The Elastic Region of Unloading</u>

From (5.2.16), (5.2.17), (5.2.18) and (5.2.19) follows that the stress, strain and displacement in the elastic region of unloading may be written as

$$\tau_y(x,t) = -\frac{\alpha E_1}{1-\nu_1}\left[T(x,t) - T_m \frac{a}{x}\right] - 2y_o \tag{5.3.4}$$

$$\epsilon_x(x,t) = \frac{1+\nu_1}{1-\nu_1}\alpha\, T(x,t) + \left(3\alpha - \frac{1+\nu_1}{1-\nu_1}\alpha\right)\left[T_m \frac{a}{x} - \frac{2(1-\nu_1)}{E_1\alpha} y_o\right] \tag{5.3.5}$$

Both (5.3.4) and (5.3.5) are applicable under the conditions

$$\rho_2 \geqslant x \geqslant a \quad , \quad t_{pa} > t \geqslant t_m \quad ,$$

or

$$\rho_2 \geqslant x \geqslant \rho_3 \quad , \quad t > t_{pa} \quad .$$

$$U(x,t) = \frac{1+\nu_1}{1-\nu_1} a\,\alpha\, T_m \left(\frac{\pi}{2} e\right)^{1/2}\left[\psi\left(\frac{x}{2\sqrt{\kappa t}}\right) - 1\right]$$

$$+ \left(3\alpha - \frac{1+\nu_1}{1-\nu_1}\alpha\right)\left\{ a\, T_m \left(\frac{\pi}{2} e\right)^{1/2}\left[\psi\left(\frac{\rho_2}{2\sqrt{\kappa t}}\right) - \left(\frac{\rho_1}{2\sqrt{\kappa t}}\right)\right]\right.$$

$$\left. + \frac{2(1-\nu_1)}{E_1\alpha} y_o (\rho_1 - x) + T_m\, a \ln\frac{x}{\rho_2}\right\}$$

for

$$\rho_2 \geqslant x \geqslant a \quad , \quad t_{pa} > t > t_m \quad ,$$

$$\rho_2 \geqslant x \geqslant \rho_3 \quad , \quad t_{mR_1} > t > t_{pa} \quad . \tag{5.3.6}$$

or

$$U(x,t) = \frac{1+\nu_1}{1-\nu_1} a \alpha T_m \left(\frac{\pi}{2} e\right)^{1/2} \left[\psi\left(\frac{x}{2\sqrt{\kappa t}}\right) - 1\right]$$

$$+ \left(3\alpha - \frac{1+\nu_1}{1-\nu_1}\alpha\right)\left[\frac{2(1-\nu_1)}{E_1\alpha} y_o \;(R_1 - x) + T_m a \ln \frac{x}{R_1} \right]$$

for $R_1 \geq x \geq a \quad , \quad t_{pa} > t > t_{mR_1} \quad ,$

$R_1 > x > \rho_3 \quad , \quad t > t_{pa} > t_{mR_1} \quad .$ (5.3.7)

where ρ_1, ρ_2, t_m, t_{mR_1}, R_1, t_{pa}, and ρ_3 in (5.3.4) to (5.3.7) are obtained from (4.4.5), (4.5.1), (4.2.4), (4.5.2), (4.4.7), (5.2.25) and (5.2.26), respectively, with the quantity B in these equations being set equal to zero.

From (5.3.4), (5.3.5) and (5.3.7) we now obtain the following results for the residual stress, strain and displacement:

$$\tau_R(x) = \frac{E_1\alpha}{1-\nu_1} T_m \frac{a}{x} - 2y_o \tag{5.3.8}$$

$$\epsilon_R(x) = \left(3\alpha - \frac{1+\nu_1}{1-\nu_1}\alpha\right) \left[T_m \frac{a}{x} - \frac{2(1-\nu_1)}{E_1\alpha} y_o\right] \tag{5.3.9}$$

$$U_R(x) = \left(3\alpha - \frac{1+\nu_1}{1-\nu_1}\alpha\right) \left[T_m a \ln \frac{x}{R_1} + \frac{2(1-\nu_1)}{E_1\alpha} (R_1 - x)\right]$$

$$- \frac{1+\nu_1}{1-\nu_1} a\alpha T_m \left(\frac{\pi}{2} e\right)^{1/2} \tag{5.3.10}$$

Equations (5.3.8), (5.3.9) and (5.3.10) are valid in the region $R_1 \geq x \geq a$ for $T_m < T_M$, or in the region $R_1 \geq x \geq R_{11}$ for $T_m > T_M$ where by (5.2.27), (5.2.23), R_{11} and T_M are expressed in the present case by

$$R_{11} = \frac{a\,T_m}{4\,y_o}\;\frac{\alpha E_1}{1-\nu_1} \tag{5.3.11}$$

$$T_M = 4\,y_o\;\frac{1-\nu_1}{\alpha E_1} \tag{5.3.12}$$

5-33 <u>The Plastic Region Formed During Unloading</u>

If the maximum boundary temperature T_m is greater than T_M given by (5.3.12), a new plastic region of unloading emerges from the boundary at the instant t_{pa}. This region eventually extends to the position $x = R_{11}$ which is given by (5.3.11). The relevant solutions for this region are found from (5.2.28), (5.2.29), (5.2.30), (5.2.21), (5.2.32), and are as follows:

$$\tau_y(x_1 t) = 2\,y_o \tag{5.3.13}$$

$$\epsilon_x(x,t) = 3\,\alpha\,T\,(x,t) + (3\,\alpha - \frac{1+\nu_1}{1-\nu_1}\alpha)\,\frac{2(1-\nu_1)}{E_1\alpha}\,y_o \tag{5.3.14}$$

$$U(x,t) = 3\,\alpha\,T_m\,a\,(\tfrac{\pi}{2}\,e)^{1/2}\,[\psi(\frac{x}{2\sqrt{\kappa t}}) - \psi(\frac{\rho_3}{2\sqrt{\kappa t}})]$$

$$+ (3\alpha - \frac{1+\nu_1}{1-\nu_1}\alpha)\;\frac{2(1-\nu_1)}{E_1\alpha}\,y_o\,(x - \rho_3) + g\,(\rho_3,t)$$

for $\quad \rho_3 \geqslant x \geqslant a$ (5.3.15)

where
$$g(\rho_3 t) = \frac{1+\nu_1}{1-\nu_1}\;a\,\alpha\,T_m\,(\tfrac{\pi}{2}\,e)^{1/2}\,[\psi(\frac{\rho_3}{2\sqrt{\kappa t}}) - 1]$$

$$+ (3\,\alpha - \frac{1+\nu_1}{1-\nu_1})\left\{ a\,T_m\,(\tfrac{\pi}{2}\,e)^{1/2}\,[\psi(\frac{\rho_2}{2\sqrt{\kappa t}}) - \psi\,(\frac{\rho_1}{2\sqrt{\kappa t}})]\right.$$

$$\left. + \frac{2(1-\nu_1)}{E_1\alpha}\,y_o\,(\rho_1 - \rho_3) + T_m\,a\,\ln\frac{\rho_3}{\rho_2}\right\} \quad \text{for} \quad t_{mR_1} > t > t_{pa}$$

(5.3.16)

or

$$g(\rho_3,t) = \frac{1+\nu_1}{1-\nu_1} \; a \, \alpha \, T_m \left(\frac{\pi}{2} e \right)^{1/2} \left[\psi\left(\frac{\rho_3}{2\sqrt{\kappa t}}\right) - 1 \right] + \left(3\alpha - \frac{1+\nu_1}{1-\nu_1}\alpha\right)$$

$$\left[T_m \, a \, \ln \frac{\rho_3}{1} + \frac{2(1-\nu_1)}{\alpha E_1} y_o \, (R_1 - \rho_3) \right]$$

for

$$t > t_{pa} > t_{mR_1} \tag{5.3.17}$$

It should be noted that ρ_3 and t_{pa} in (5.3.16) and (5.3.17) can be determined from (5.2.26) and (5.2.25) with the quantity B there being set equal to zero. Finally, the residual stress, strain and permanent deformation obtained from (5.3.13), (5.3.14), (5.3.15) and (5.3.17):

$$\tau_R(x) = 2 \, y_o \tag{5.3.18}$$

$$\epsilon_R(x) = \left(3\alpha - \frac{1+\nu_1}{1-\nu_1}\alpha\right) \frac{2(1-\nu_1)}{E_1 \alpha} y_o \tag{5.3.19}$$

$$U_R(x) = \left(3\alpha - \frac{1+\nu_1}{1-\nu_1}\alpha\right) \; T_m \, a \, \ln \frac{R_{11}}{R_1}$$

$$+ \frac{2(1-\nu_1)}{E_1 \alpha} \; y_o \; (x + R_1 - 2R_{11}) \; - \frac{1+\nu_1}{1-\nu_1} a \, \alpha \, T_m \left(\frac{\pi}{2} e \right)^{1/2} \tag{5.3.20}$$

The equations (5.3.18), (5.3.19) and (5.3.20) are applicable only in the region $R_{11} \geq x \geq a$ where R_{11} is given by (5.3.11).

Chapter 6. Elastic Plastic Response of a Laterally Constrained Plate to a Uniformly Applied Heat Pulse.

6-1 Introduction

Transient thermal stress analysis of free plates has been treated by Weiner, Landau, Zwicky [6.1] [6.2] [6.3], Yushel [6.4], and most recently by Mendelson and Spero [6.5]. In this chapter, we shall consider the response of an infinite plate constrained in the lateral direction, and subjected to a uniformly applied heat pulse at one boundary, the other boundary being held fixed and either maintained at zero temperature or insulated.

The purpose of studying a plate of finite thickness is to remove a basic shortcoming of the solution found for the half-space, namely, the absence of a characteristic dimension of length. Specifically, we shall relate the dimension of the plastically deformed region to the plate thickness and thus establish a basis for obtaining approximate solutions of plate problems from the corresponding solution for the half-space.

If the mechanical and thermal boundary conditions for a plate problem are similar to those applied for the half-space, then the analysis will also be, in essence, identical with that carried out for the half-space. In fact, the specific solution derived in Chapters 4 and 5 also apply directly for plates, provided that the temperature distribution for the half-space is replaced by the appropriate temperature distribution in the plate. Therefore, let us first investigate the temperature solutions for plate problem.

6-2 The Temperature Problems

The temperature distribution in an infinite plate, produced by a heat pulse applied uniformly at one boundary, the other boundary being maintained at zero temperature, may be obtained by superposing the temperature fields induced by a suitably located instantaneous source and sink.

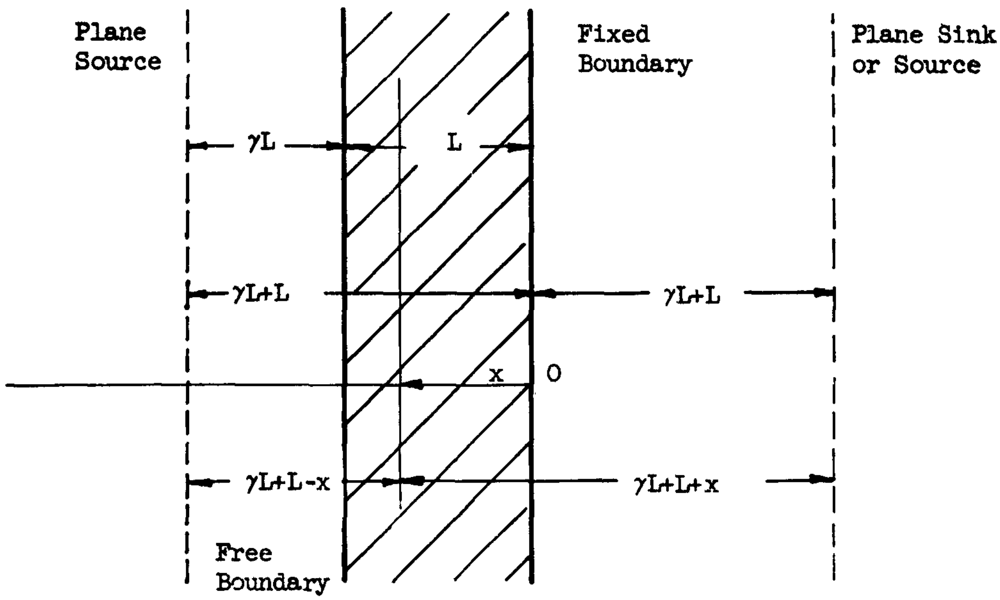

Fig. 6-1. Location of Source and Sink for a Plate Subjected to a Uniformly Applied Heat Pulse at One Boundary, With the Other Boundary Either Maintained at Zero Temperature or Insulated.

Let the plate be initially at a uniform temperature T_o . We characterize the origin of coordinates as shown in Fig. 6-1, and assume that the boundary $x = o$ is fixed while the other boundary at $x = L$ remains free. An instantaneous plane source and sink of equal strength are placed symmetrically to the $x = o$ plane; due to the symmetry of these locations, the temperature change at $x = o$ is zero. At the free boundary, or at any other plane between the two boundaries, the temperature will increase from zero to a maximum and again decrease to zero.

The superposition of temperature solutions due to a source and a sink to describe the increase of temperature over the reference temperature T_o is thus expressed by

$$T(x,t) = C_1 \frac{1}{t^{1/2}} \left[e^{-\frac{(\gamma L+L-x)^2}{4\kappa t}} - e^{-\frac{(\gamma L+L+x)^2}{4\kappa t}} \right] . \tag{6.2.1}$$

Here C_1 is a constant, L denotes the thickness of the plate, and γL represents the distance of the source from the free boundary (Fig. 6-1), the quantity γ being an arbitrary constant. By varying γ, the shape of the heat pulse, or, equivalently, the time rate of increase of temperature may be adjusted. By inspection of (6.2.1) we find that $T = o$ for $x = o$. If two sources of equal strength are placed symmetrically to the $x = o$ plane, then no heat will be transferred across this plane, and the temperature distribution will correspond to the case when the plate is subjected to a uniformly applied heat pulse at the free boundary while the fixed boundary is insulated (Fig. 6-1). We find

$$T(x,t) = C_2 \frac{1}{t^{1/2}} \left[e^{-\frac{(\gamma L+L-x)^2}{4\kappa t}} + e^{-\frac{(\gamma L+L+x)^2}{4\kappa t}} \right] \tag{6.2.2}$$

where C_2 is a constant.

The maximum temperature T_{mx} attainable at an arbitrary plane x in the plate, and the corresponding instant t_{mx} at which the temperature assumes this value T_{mx} are very significant in the analysis of the mechanical response. In order to evaluate T_{mx}, it is first necessary to determine t_{mx} from the condition

$$\left.\frac{\partial T}{\partial t}\right)_x = o \tag{6.2.3}$$

The solutions appropriate to (6.2.1) and (6.2.2), respectively, are then given by

$$t_{mx} = \frac{\left[e^{-\frac{(\gamma L+L-x)^2}{4\kappa t_{mx}}} \frac{(\gamma L+L-x)^2}{2\kappa} - e^{-\frac{(\gamma L+L+x)^2}{4\kappa t_{mx}}} \frac{(\gamma L+L+x)^2}{2\kappa}\right]}{e^{-\frac{(\gamma L+L-x)^2}{4\kappa t_{mx}}} - e^{-\frac{(\gamma L+L-x)^2}{4\kappa t_{mx}}}}$$

for

$$L \geqslant x > o \tag{6.2.4}$$

and

$$t_{mx} = \frac{\left[e^{-\frac{(\gamma L+L-x)^2}{4\kappa t_{mx}}} \frac{(\gamma L+L-x)^2}{2\kappa} + e^{-\frac{(\gamma L+L+x)^2}{4\kappa t_{mx}}} \frac{(\gamma L+L+x)^2}{2\kappa}\right]}{e^{-\frac{(\gamma L+L-x)^2}{4\kappa t_{mx}}} + e^{-\frac{(\gamma L+L+x)^2}{4\kappa t_{mx}}}}$$

for

$$L \geqslant x \geqslant o \tag{6.2.5}$$

The constants C_1 and C_2 in (6.2.1) and (6.2.2) may be eliminated by introducing the maximum values T_{mL}, ordinarily assumed to be known, for the plane $x = L$. Denoting by t_{mL} the instant for which $T = T_{mL}$, we obtain from (6.2.1), (6.2.4), (6.2.2), (6.2.5) the results

$$C_1 = T_{mL} \frac{\left[e^{-\frac{(\gamma L)^2}{4\kappa t_{mL}}} \frac{(\gamma L)^2}{2\kappa} - e^{-\frac{(2L+\gamma L)^2}{4\kappa t_{mL}}} \frac{(2L+\gamma L)^2}{2\kappa}\right]^{1/2}}{\left[e^{-\frac{(\gamma L)^2}{4\kappa t_{mL}}} - e^{-\frac{(2L+\gamma L)^2}{4\kappa t_{mL}}}\right]^{3/2}} \tag{6.2.6}$$

and

$$C_2 = T_{mL} \frac{\left[e^{-\frac{(\gamma L)^2}{4\kappa t_{mL}}} \frac{(\gamma L)^2}{2\kappa} + e^{-\frac{(\gamma L+2L)^2}{4\kappa t_{mL}}} \frac{(2L+\gamma L)^2}{2\kappa}\right]^{1/2}}{\left[e^{-\frac{(\gamma L)^2}{4\kappa t_{mL}}} + e^{-\frac{(2L+\gamma L)^2}{4\kappa t_{mL}}}\right]^{3/2}} \tag{6.2.7}$$

where t_{mL} in (6.2.6) and (6.2.7) is found from (6.2.4) and (6.2.5), respectively.

Having derived expressions for t_{mx} , C_1 and C_2 , we are now in a position to calculate the maximum temperature T_{mx} attainable at any point in the plate. The variations of temperature T with time t for various values of x for both solutions (6.2.1) and (6.2.2) have been plotted in terms of dimensionless variables T/T_{mL} , x/L , and $t/L^2/2\kappa$, and are shown in Figs. 6-2 and 6-3, respectively.

6-3 Response of a Laterally Constrained Plate to a Uniformly Applied Heat Pulse at Its Free Boundary With the Fixed Boundary Maintained at Zero Temperature

The plate considered in this section is one constrained in the lateral directions, with traction free boundaries $x = o$ and $x = L$.

The solution

$$U_{xx} = U(x,t) \qquad U_{yy} = U_{zz} = o \tag{6.3.1}$$

$$\epsilon_{xx} = \epsilon\ (x,t) \qquad \epsilon_{yy} = \epsilon_{zz} = o \tag{6.3.2}$$

$$\tau_{yy} = \tau_{zz} = \tau \qquad \tau_{xx} = o \tag{6.3.3}$$

then satisfies both the stress equation of equilibrium and the traction boundary conditions. A further boundary condition is imposed on the displacements by

$$U_{xx} = o \quad \text{at} \quad x = o \quad \text{for} \quad t > o \tag{6.3.4}$$

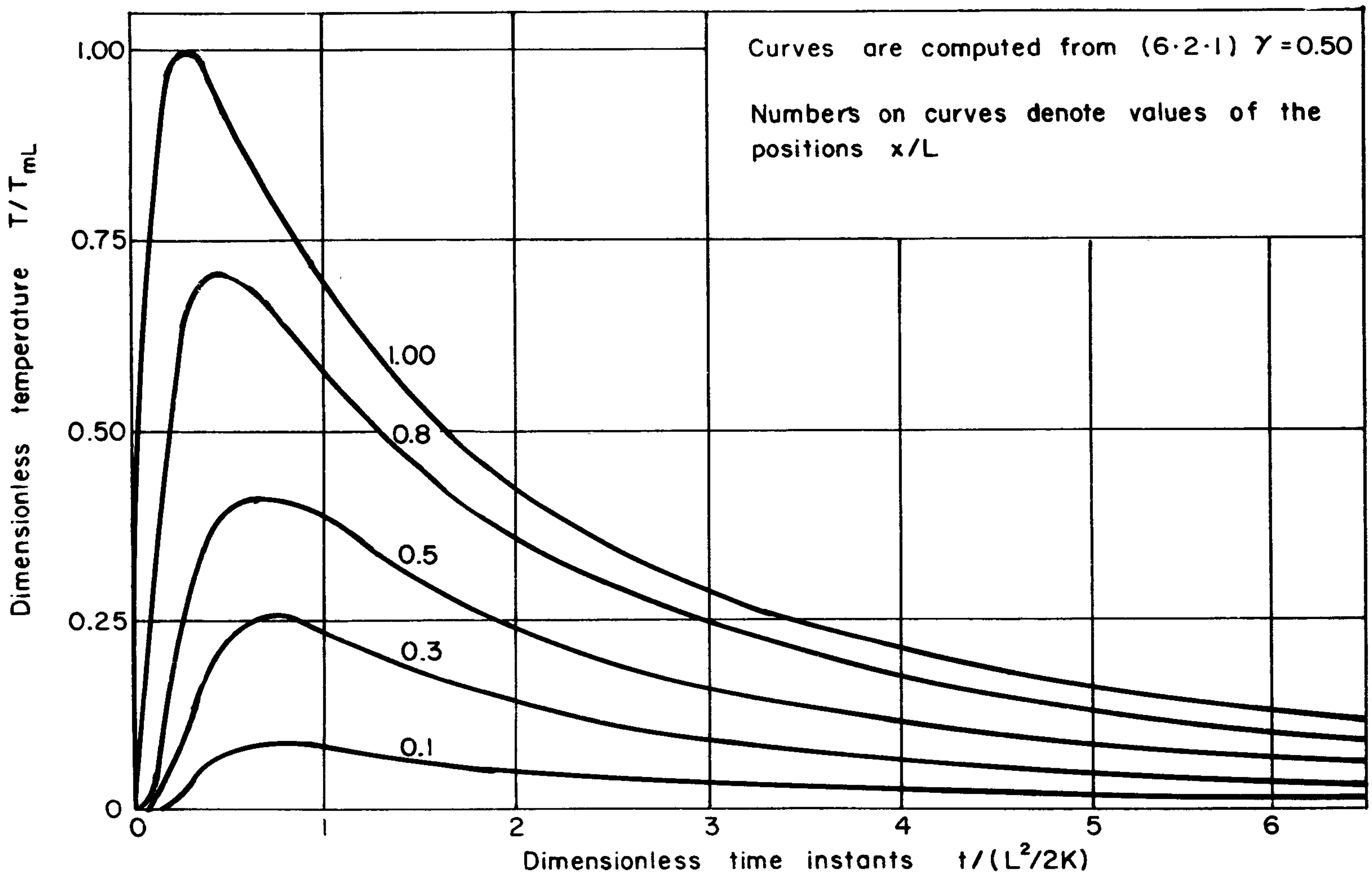

FIG 6-2 TEMPERATURE CHANGE WITH TIME AT DIFFERENT POSITIONS

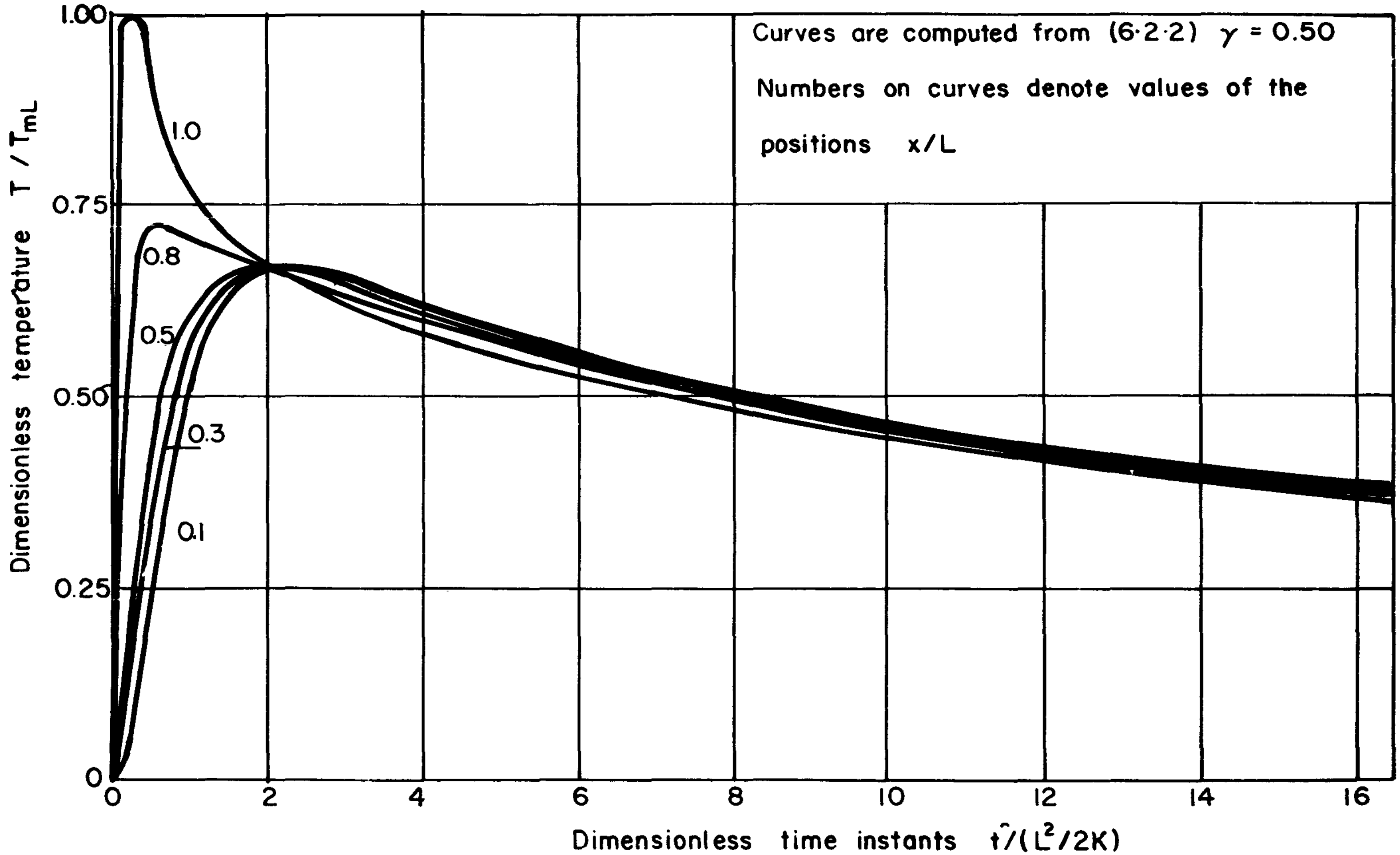

FIG. 6-3 TEMPERATURE CHANGE WITH TIME AT DIFFERENT POSITIONS

The similarity between the problem for a plate and the half-space becomes evident from inspection of equations (4.3.5) to (4.3.7). In particular, comparison of the temperature distribution represented by Fig. 4-2 and Fig. 6-2, and the fact that the temperature and displacement in the half-space were assumed to vanish for large x, bring out clearly the analogy between the two problems. Therefore the response of the plate to the uniformly applied heat pulse represented by (6.2.1) is, in essence, the same as that of the half-space. Especially, if the proper temperature functions are used, the equations describing the transient and residual stresses and strains in the half-space may be used to describe the corresponding quantities in the plate.

During the initial stage of the heat pulse, the entire plate is elastic. The non vanishing stress and strain components are given by (4.3.9) and (4.3.8):

$$\tau = \tau_{yy} = \tau_{zz} = -\frac{\alpha E_1}{1-\nu_1} T \qquad (4.3.9)\ \text{Re}$$

$$\epsilon = \epsilon_{xx} = \frac{1+\nu_1}{1-\nu_1} \alpha T \qquad (4.3.8)\ \text{Re}$$

The maximum shearing stress is represented by

$$q = -\frac{\tau_{yy}}{2} = \frac{\alpha E_1}{2(1-\nu_1)} T \qquad (4.3.12)\ \text{Re}$$

where the temperature T for (4.3.9) Re, (4.3.8) Re and (4.3.12) is given by (6.2.1).

The displacement U_{xx} is obtained by integrating (4.3.8) Re in conjunction with (6.2.1). In view of the condition (6.3.4) we obtain the result

$$U_{xx}(x,t) = \frac{1+\nu_1}{1-\nu_1}\ \alpha\ c_1\ (\pi\kappa)^{1/2}\ [2\psi(\frac{\gamma L+L}{2\sqrt{\kappa t}}) - \psi(\frac{\gamma L+L-x}{2\sqrt{\kappa t}}) - \psi(\frac{\gamma L+L+x}{2\sqrt{\kappa t}})] \tag{6.3.5}$$

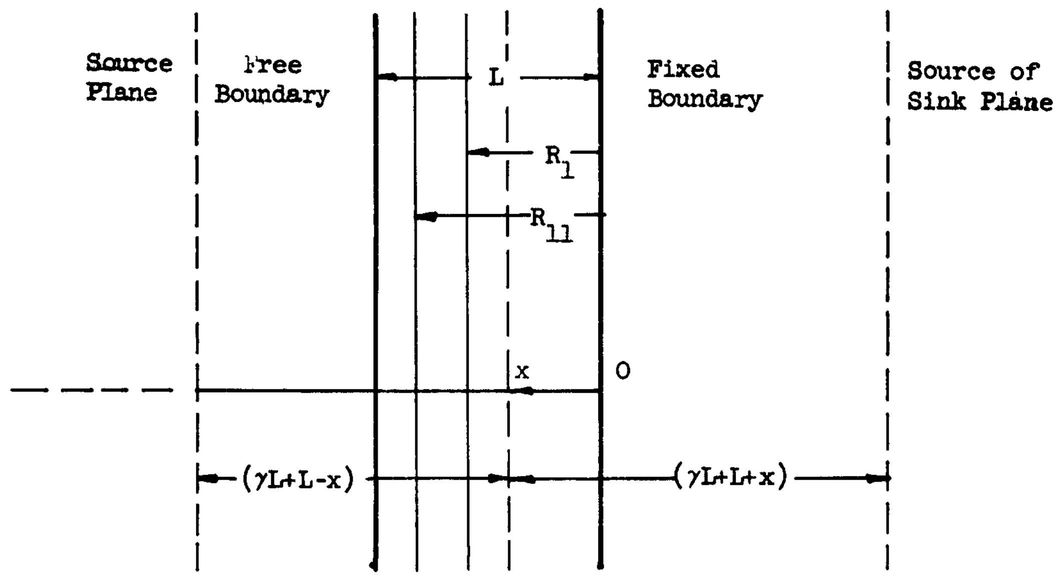

Fig. 6-4 Boundaries of the Total Plastically Deformed and Steady State Plastic Regions in the Plate.

It is seen from Figs. 6-1 and 6-4 that the origin of coordinates is chosen at the fixed boundary of the plate instead of at the source plane as was done in the case of the half-space (Fig. 4-10). Since the heat pulse is applied at the plane $x = L$, the dimension of the total plastically deformed region and that of the steady state plastic region are given by $(L - R_1)$ and $(L - R_{11})$, respectively. Here R_1 and R_{11} still denote the distances between the origin and the boundary of the corresponding regions. In order to estimate the damage produced by the heat pulse, the location of the boundary of these two regions will now be determined. By the criterion of Tresca, yielding occurs wherever the maximum shearing stress becomes equal to the yield stress in simple shear. From examination of (4.3.12), (6.2.1) and Fig. 6-2, it is evident that the initial yielding starts at the boundary $x = L$ (Fig. 6-4), and that the elastic plastic interface will progress into the plate from this boundary as the temperature continues to increase. Following the same reasoning as that presented in section 4-4, the boundary of the total plastically deformed region is thus determined by solving the following two simultaneous algebraic equations:

$$t_{mR_1} = \frac{\left[e^{-\frac{\gamma L+L-R_1}{4\kappa t_{mR_1}}} \frac{(\gamma L+L-R_1)^2}{2\kappa} - e^{-\frac{(\gamma L+L+R_1)^2}{4\kappa t_{mR_1}}} \frac{(\gamma L+L+R_1)^2}{2\kappa}\right]}{e^{-\frac{(\gamma L+L-R_1)^2}{4\kappa t_{mR_1}}} - e^{-\frac{(\gamma 1+L+R_1)^2}{4\kappa t_{mR_1}}}} \tag{6.3.6}$$

and

$$\frac{y_o}{B + E_1\alpha/2(1-\nu_1)} = T_{mR_1} \tag{6.3.7}$$

where

$$T_{mR_1} = C_1 \frac{1}{t_{mR_1}^{1/2}} \left[e^{-\frac{(\gamma L+L-R_1)^2}{4\kappa t_{mR_1}}} - e^{-\frac{(\gamma L+L+R_1)^2}{4\kappa t_{mR_1}}}\right]$$

Here, it may be noted that (6.3.7) is essentially identical to (4.4.6).

Simple and explicit expression for R_1 in terms of other known quantities cannot be found from (6.3.6) and (6.3.7); therefore, the following graphical method is suggested (Fig. 6-5).

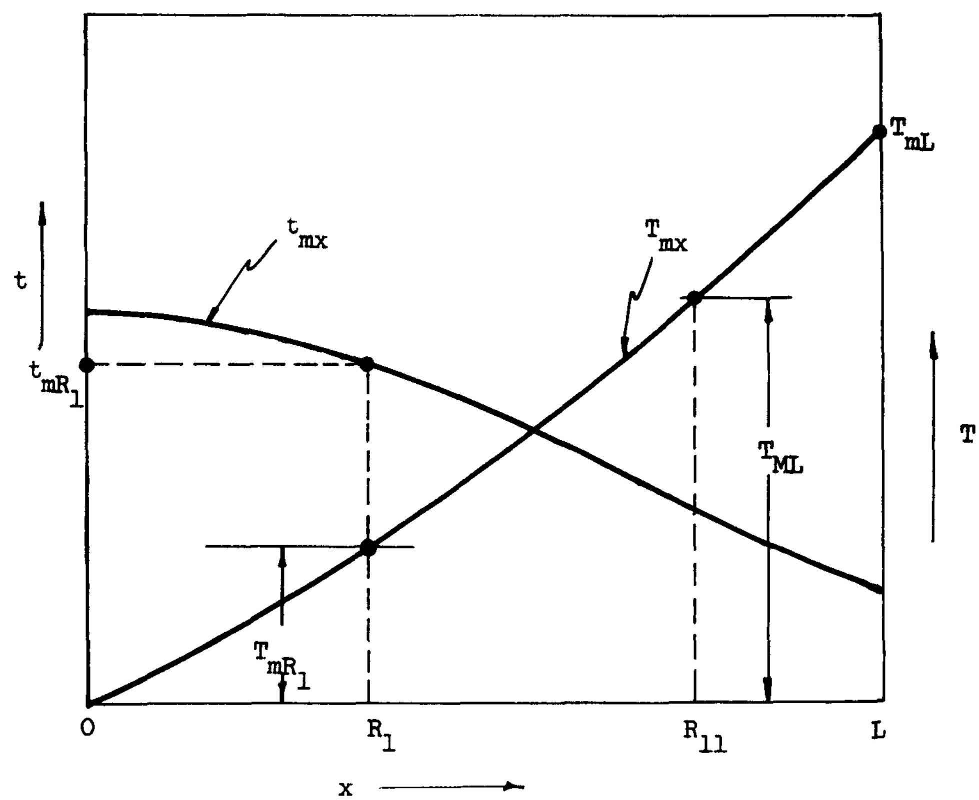

Fig. 6-5 Graphical Determination of R_1, t_{mR_1} and R_{11}

By the use of (6.2.1) and (6.2.4), the corresponding values of t_{mx} and T_{mx} for a given value of x may be calculated, and plotted as shown in Fig. 6-5. If a horizontal of height equal to T_{mR_1} and given by (6.3.7) is drawn, then the coordinate of its intersection with the T_{mx} curve gives both t_{mR_1} and R_1. In the case when the maximum boundary temperature T_{mL} is greater than T_{ML} for which plastic flow occurs during unloading, a new plastic region would be formed in the region of elastic unloading and the steady state position R_{11} of this plastic elastic interface may be determined by locating in the plastically deformed region the plane at which the maximum temperature attainable is equal to T_{ML}. We obtain the solutions for the plate in the plastic region formed during unloading by replacing in (4.6.5), (4.6.6) the temperature solution appropraite to the half-space problem by (6.2.1). In particular, replacing the maximum temperature $T_m\, a/x$ by T_{ML} in (4.6.2), we find that

$$T_{ML} = \frac{1}{M + \frac{\alpha E_1}{2(1-\nu_1)} - B}\left[2\, y_o + \frac{y_o}{B + \frac{\alpha E_1}{2(1-\nu_1)}}(M-2B)\right] \tag{6.3.8}$$

By drawing in Fig. 6-5 a horizontal of height equal to T_{ML} given by (6.3.8), the value of R_{11} can be read off directly for any value of γ from the point of intersection of this horizontal with the T_{mx} curve. This process is equivalent to the simultaneous solution of the following two equations for R_{11} and $t_{mR_{11}}$:

$$C_1 \frac{1}{t_{mR_{11}}}\left[e^{-\frac{(\gamma L+L-R_{11})^2}{4\kappa t_{mR_{11}}}} - e^{-\frac{(\gamma L+L+R_{11})^2}{4\kappa t_{mR_{11}}}}\right]$$

$$= \frac{1}{M + \frac{\alpha E_1}{2(1-\nu_1)} - B}\left[2\, y_o + \frac{y_o}{B + \frac{\alpha E_1}{2(1-\nu_1)}}(M-2B)\right]$$

(6.3.9)

and

$$t_{mR_{11}} = \frac{[e^{-\frac{(\gamma L+L-R_{11})^2}{4\kappa t_{mR_{11}}}} \frac{(2L+L-R_{11})^2}{2\kappa} - e^{-\frac{(\gamma L+L+R_{11})^2}{4\kappa t_{mR_{11}}}} \frac{(\gamma L+L+R_{11})^2}{2\kappa}]}{e^{-\frac{(\gamma L+L-R_{11})^2}{4\kappa t_{mR_{11}}}} - e^{-\frac{(\gamma L+L+R_{11})^2}{4\kappa t_{mR_{11}}}}} \tag{6.3.10}$$

The quantity C_1 in (4.3.9) is given by (6.2.6). With R_1 and R_{11} determined, the position of the steady state elastic plastic interface and the dimension of the total plastically deformed region are known. The remaining steps in the plate problem are entirely analogous to those of the half-space problem, and, therefore, will not be elaborated here.

6-4 The Response of a Laterally Constrained Plate to a Heat Pulse Uniformly Applied at its Free Boundary with the Fixed Boundary Insulated

The elastic solution for the present case is the same as that presented in the last section, except that the temperature function T in (4.3.9), (4.3.8) and (4.3.12) is now given by (6.2.2). The counterpart of (6.3.5) then reads

$$U(x,t) = \frac{1+\nu_1}{1-\nu_1} \alpha C_2 (\pi\kappa)^{1/2} [\psi(\frac{\gamma L+L+x}{2\sqrt{\kappa t}}) - \psi(\frac{\gamma L+L-x}{2\sqrt{\kappa t}})]$$

$$\text{for} \quad L \geq x \geq 0 \tag{6.4.1}$$

If the temperature T_{mR_1} given by (6.3.7) is greater than T_{mo}, which is defined as the maximum temperature attainable at the fixed and insulated boundary, the position of the total plastically deformed region R_1 may be obtained from the solution of the simultaneous equations

$$C_2 \frac{1}{t_{mR_1}^{1/2}} \left[e^{-\frac{(\gamma L+L-R_1)^2}{4\kappa t_{mR_1}}} + e^{-\frac{(\gamma L+L+R_1)^2}{4\kappa t_{mR_1}}} \right]$$

$$= \frac{y_o}{B + \frac{E_1 \alpha}{2(1-\nu_1)}} \tag{6.4.2}$$

and

$$t_{mR_1} = \frac{\left[e^{-\frac{(\gamma L+L-R_1)^2}{4\kappa t_{mR_1}}} \frac{(\gamma L+L-R_1)^2}{2\kappa} + e^{-\frac{(\gamma L+L+R_1)^2}{4\kappa t_{mR_1}}} \frac{(\gamma L+L+R_1)^2}{2\kappa} \right]}{e^{-\frac{(\gamma L+L-R_1)^2}{4\kappa t_{mR_1}}} + e^{-\frac{(\gamma L+L+R_1)^2}{4\kappa t_{mR_1}}}} \tag{6.4.3}$$

where C_2 in (6.4.2) is given by (6.2.7). The graphical method suggested in the last section for locating R_1 is still applicable here. If on the other hand, T_{mR_1} is less than T_{mo}, the entire plate would experience varying amounts of plastic loading and would become elastic again once the temperature begins to decrease.

The maximum boundary temperature T_{ML} for the present case is also given by (6.3.8).

The steady state position R_{11} of the elastic plastic interface in the plate for maximum boundary temperature T_{mL} greater than T_{ML} and for a given value of γ is obtained from the solution of the simultaneous equations

$$C_2 \frac{1}{t_{mR_{11}}^{1/2}} \left[e^{-\frac{(\gamma L+L-R_{11})^2}{4\kappa t_{mR_{11}}}} + e^{-\frac{(\gamma L+L+R_{11})^2}{4\kappa t_{mR_{11}}}} \right]$$

$$= \frac{1}{M + \frac{\alpha E_1}{2(1-\nu_1)} - B} \left[2\, y_o + \frac{y_o}{B + \frac{E_1 \alpha}{2(1-\nu_1)}} (M-2B) \right] \tag{6.4.4}$$

and

$$t_{mR_{11}} = \frac{\left[e^{-\frac{(\gamma L+L-R_{11})^2}{4\kappa t_{mR_{11}}}} \frac{(\gamma L+L-R_{11})^2}{2\kappa} + e^{-\frac{(\gamma L+L+R_{11})^2}{4\kappa t_{mR_{11}}}} \frac{(\gamma L+L+R_{11})^2}{2\kappa} \right]}{e^{-\frac{(\gamma L+L-R_{11})^2}{4\kappa t_{mR_{11}}}} + e^{-\frac{(\gamma L+L+R_{11})^2}{4\kappa t_{mR_{11}}}}} \tag{6.4.5}$$

where C_2 in (6.4.4) is given by (6.2.7). The value of R_{11} can also be determined graphically as was indicated in the last section.

The transient and residual solutions for the plate considered here can be derived in the same manner as was done in Chapters 4 and 5. No additional challenge is posed by these problems, hence, further analysis is not pursued.

Chapter 7 Numerical Results and Discussion

7-1 Introduction and General Discussion

A shortcoming of the half-space problem is that it lacks a characteristic dimension of length; in order to remove this shortcoming a plate of finite thickness was treated in Chapter 6. We now consider the question of whether it is possible to predict the transient and residual stresses and deformations in a plate from the solution of the half-space problem. This question is considered in some detail in the following pages.

The analogy of problems for plates and the half-space was previously referred to in Chapter 6, and it was noted there that the equations for stresses and strains in the half-space problem may be used for the plate problems if the temperature function is appropriately modified. Therefore, it is of interest to determine the extent to which special results of the half-space problem may be directly applied to plate problems.

In the case that the effect of the dependence of yield stress on temperature dominates over the strain hardening effect of the material, the maximum transient stress attainable in the half-space is found from (4.3.8), (4.4.4), (4.4.6) to be dependent only on material properties, and is equal to twice the initial yield stress in shear. The maximum transient stress for the plate is therefore given by

$$\tau_{max} \doteqdot -2[y_o - BT_1] \tag{7.1.1}$$

where
$$T_1 = T_{mR_1} = \frac{y_o}{B + \alpha E_1/2(1-\nu_1)}$$

A close inspection of (4.5.12), (4.6.11) describing residual stress, and of (4.4.27), (4.5.13), (4.6.11) describing transient and residual strains, reveals that these quantities depend in addition to material properties, solely on the maximum temperature attainable at a particular position, and assume their maximum values at the boundary $x = a$. The highest temperature T_m attainable in the half-space, and the highest temperature T_{mL} attainable in the plate are ordinarily assumed to be known. If T_m is set equal to T_{mL}, the maximum transient strain, maximum residual stress and strain of the half-space become equal to the corresponding quantities of the plate. Therefore, in view of the foregoing remarks, and by (4.4.27), the maximum transient strain in the plate may be expressed in the form

$$\epsilon_{max} = N\,T_m - [(N - \frac{1+\nu_1}{1-\nu_1}\,\alpha)\,\frac{y_o}{B + \alpha\,E_1/2(1-\nu_1)}] \qquad (7.1.2)$$

whereas, by (4.5.12), (4.5.13), the maximum residual stress and strain are obtained from

$$\tau_{R_{max}} = (M + \frac{\alpha E_1}{1-\nu_1})\,T_{mL} - [2y_o + \frac{y_o}{B + \alpha E_1/(2(1-\nu_1)}\,(M-2B)]\,,$$

$$(7.1.3)$$

$$\epsilon_{R_{max}} = (N\,\frac{1+\nu_1}{1-\nu_1}\,\alpha)\,[T_{mL} - \frac{y_o}{B + \alpha E_1/2(1-\nu_1)}]\,,$$

for $\quad T_{mL} < T_{ML}$.

Here T_{ML} is the maximum free boundary temperature for which plastic flow in the plate during unloading will impend. For the case when $T_{mL} > T_{ML}$, it follows from (4.6.11), (4.6.12) that

$$\tau_{R_{max}} = [(M + \frac{\alpha E_1}{1-\nu_1}) - \frac{M' + \alpha E_1/1-\nu_1}{2B - \alpha E_1/1-\nu_1}(2B - \frac{\alpha E_1}{1-\nu_1} - 2M)] T_{mL}$$

$$- (2\frac{M' + \alpha E_1/1-\nu_1}{2B - \alpha E_1/1-\nu_1} + 1)[2y_o + \frac{y_o}{B + \alpha E_1/2(1-\nu_1)}(M-2B)]$$

$$\epsilon_{R_{max}} = [(N - \frac{1+\nu_1}{1-\nu_1}\alpha) - \frac{N' - 1+\nu_1/1-\nu_1}{2B - \alpha E_1/1-\nu_1}(2B - \frac{\alpha E_1}{1-\nu_1} - 2M)] T_{mL} \tag{7.1.4}$$

$$- \left\{\frac{N' - (1+\nu_1/1-\nu_1)\alpha}{2B - \alpha E_1/1-\nu_1}[4 y_o + \frac{2 y_o}{B + \alpha E_1/2(1-{}_1)}(M-2B)]\right.$$

$$\left. + \frac{N - (1+\nu_1/1-\nu_1)\alpha}{B + \alpha E_1/2(1-\nu_1)} y_o\right\}$$

It is thus seen that the maximum transient and residual stresses and strain induced in a plate by a heat pulse associated with the maximum boundary temperature T_{mL} may be predicted by computing the corresponding quantities for the half-space subjected to a pulse of maximum value T_{mL}. We also note that the shapes of the two pulses need not be the same, but that their maximum amplitude must be equal in order for the similarity to exist.

Equations (7.1.1) to (7.1.4) together with (6.3.6), (6.3.7), (6.3.9), (6.3.10), (6.4.2), (6.4.3), (6.4.4), (6.4.5) for the determination of the positions R_1 and R_{11} of the boundary of the total plastically deformed and steady state plastic regions may be considered to represent the significant parts of the transient and steady state solutions for the plate problems.

It is also possible to obtain the residual stress and strain at any other place in the plastically deformed region of a plate by replacing T_{mL} in (7.1.3) or (7.1.4) by the corresponding maximum temperature T_{mx} attainable at that position. Since the displacement boundary conditions for the two problems are not exactly identical, the solutions for displacement will differ, in essence, by a rigid translation.

We adopt the quantities

$$T/T_m,\ x/a,\ \rho_1/a,\ \rho_2/a,\ \rho_3/a,\ R_1/a,\ R_{11}/a,\ U/a\ \alpha\ T_m,\ t/a^2/2\kappa$$

to represent dimensionless temperature T, position x, interface positions ρ_1, ρ_2, ρ_3, R_1, R_{11}, displacement U and time instant t, respectively, for the half-space problem. Here a is the distance between the source plane and the boundary of the half-space; for convenience, the dimensionless time is often denoted by n.

Similarly, the quantities

$$T/T_{mL},\ x/L,\ R_1/L,\ R_{11}/L$$

are adopted to represent the dimensionless temperature T, position x and steady state interface positions R_1 and R_{11} for the plate problems. The dimensionless stress and strain for both cases are expressed by τ/y_0 and $\epsilon/\alpha T$, respectively.

A detailed calculation of stresses and deformations in a half-space, and for specific data, will enable us to gain an insight into the general nature of the thermomechanical response. In view of the established similarity with plate problems, we shall also be in a position to draw conclusion regarding the behavior of plates.

An aluminum alloy (25ST) having the following material properties was selected for numerical calculation:

$$
\begin{aligned}
&E_1 = 10 \times 10^6 \text{ psi} , \qquad \alpha = 14.5 \times 10^{-6} \text{ in/in/F}^\circ , \\
&\kappa = 0.133 \text{ in}^2/\text{sec.} , \qquad \nu_1 = 0.275 , \\
&\nu_p = 0.35 , \qquad \mu_p = 0.2891 \times 10^6 \text{ psi} , \\
&\mu_E = 3.9215 \times 10^6 \text{ psi}, \\
&y = y_o - BT ,
\end{aligned}
\tag{7.1.5}
$$

where $y_o = 23{,}000$ psi , $B = 0.293 \times 10^2$ psi/F$^\circ$.

The data (7.1.5) represent the thermal and mechanical properties of an elastic, linearly strain hardening medium having a yield stress in shear which varies linearly with temperature.

In order to compare the solutions presented in chapters 4 and 5, a perfectly plastic medium, with thermal and mechanical properties identical to those given in (7.1.5) was also considered, except that there the plastic shear modulus μ_p and the coefficient B were set equal to zero.

In presenting numerical results, solid curves have been used for plotting results pertaining to the strain hardening material, whereas the corresponding plots for the perfect plastic medium are indicated by broken curves.

7-2 Numerical Results and Discussion for Plate Problems and for the Half-Space Problem

The damage incurred in a half-space or plate by a heat pulse may be characterized by the magnitudes of residual stresses and the dimension of the total plastically deformed region. As far as the residual stresses are concerned, the location and the largest magnitude of the residual stress are of great importance, and will be considered first. We recall from the discussion presented in the preceding section that the maximum residual stress occurs at the free boundaries of both a half-space and a plate. Moreover, the largest residual stresses in a half-space and a plate are equal if the associated maximum boundary temperatures T_m and T_{mL} are the same. The maximum residual stresses were calculated for different values of T_m or T_{mL} for both a strain hardening material having a temperature dependent yield stress and a perfectly plastic material having a constant yield stress. This calculation was based on the formulas (4.5.12), (7.1.3), (5.3.8), or (4.6.11), (7.1.4), (5.3.18), depending on whether plastic flow occurs during unloading or not. The results of the calculation for values of T_m or T_{mL} ranging from 175°F to 700°F are plotted in Fig. 7-1. The critical temperatures T_1 and T_{mR_1} for which plastic flow impends in a half-space and a plate, respectively, were calculated from (4.4.6), (6.3.7) and the material properties given in (7.1.5). The temperature T_M specifying incipient plastic flow in unloading was calculated either from (4.6.3) or from (6.3.7). It was found that

$$T_1 = T_{mR_1} = 178^{\circ}F \quad , \quad T_M = T_{ML} = 375.06^{\circ}F \quad , \tag{7.2.1}$$

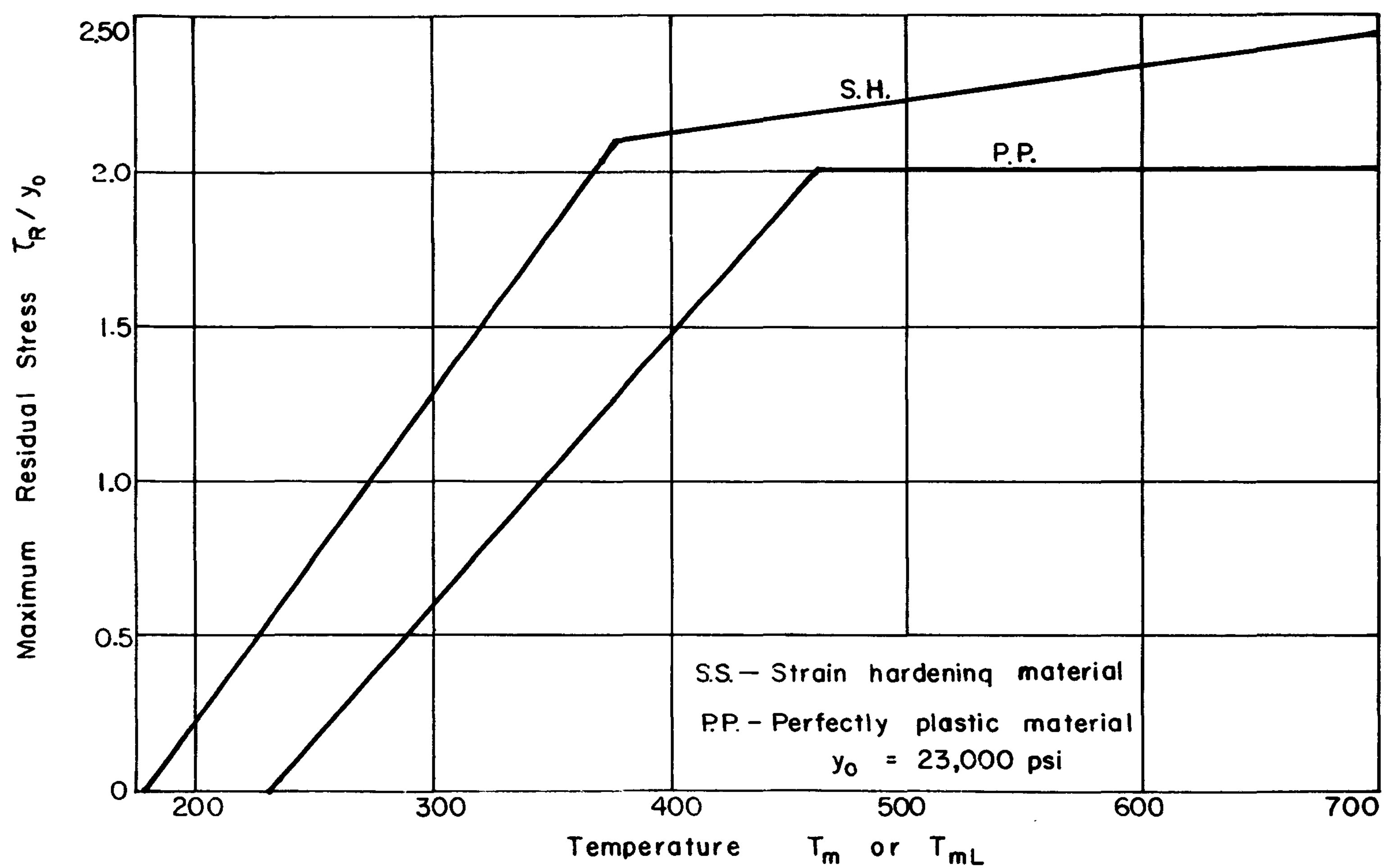

FIG 7-1 RELATION BETWEEN MAXIMUM RESIDUAL STRESS AND TEMPERATURE IN HALE SPACE OR PLATE.

for the strain hardening material, and

$$T_1 = T_{mR_1} = 230^{\circ}F \quad , \quad T_M = T_{ML} = 460^{\circ}F \quad , \tag{7.2.2}$$

for the perfectly plastic material.

We note from Fig. 7-1 that the maximum residual stress is linearly related to the maximum temperature T_m or T_{mL}. Actually, the linearity between the maximum residual stress and the maximum temperature can be readily observed on inspection of (4.5.12), (4.6.11) or (7.1.3), (7.1.4). In view of (7.2.1) and (7.2.2), we further note from Fig. 7-1 that the maximum residual stress for the strain hardening material increases more rapidly with the maximum temperature T_m for values of T_m less than T_M than for values of T_m greater than T_M. Here T_M corresponds to the maximum temperature for which incipient plastic flow during unloading impends. If the maximum temperature T_m is greater than the critical value T_M, the material is in the plastic state instead of in the elastic state as being characterized by values of T_m less than the critical value T_M. This accounts for the fact that the curves relating the maximum residual stress and maximum temperature change their slopes at $T_m = T_M$. The foregoing remarks make it clear that the maximum residual stress for the perfectly plastic material would remain constant for T_m greater than the critical value T_M (Fig. 7-1).

The dimension of the total plastically deformed region is also a characteristic measure of the damage incurred in a half-space or plate by a heat pulse. Therefore, we now proceed to calculate the position

R_1 of the boundary of the total plastically deformed region. The positions R_1 and R_{11} of the total plastically deformed and steady state plastic regions in a half-space, and corresponding to different values of T_m may be calculated from (4.4.7), (4.6.4) and (7.1.5). In order to determine R_1 and R_{11} in a plate, the steps suggested in section 6-3 were followed. The dimensionless values of R_1 for γ equal to 0.5 are plotted in Fig. 7-2 both for a half-space and a plate, γ being a parameter introduced in the temperature functions (6.2.1) and (6.2.2) to determine the shape of the heat pulse. For a plate, γL represents the distance between the source plane and the free boundary of the plate (Fig. 6-1). Similar plots showing the variation of both the position R_1 and R_{11} with maximum boundary temperature T_{mL} for the individual plates for values of γ equal to 0.50 and 1.00 are presented in Figs. 7-3 and 7-4.

In the case of the plate having a fixed and insulated boundary, two critical temperatures T_{c1} and T_{c2} were found to exist. Specifically, T_{c1} is the maximum boundary temperature for which the boundary of the total plastically deformed region coincides exactly with the insulated boundary of the plate, whereas T_{c2} is the maximum boundary temperature for which the boundary of the steady state plastic region coincides with the insulated boundary of the plate. The values of T_{c1} and T_{c2} are determined by setting each of the quantities T_{mR_1}/T_{c1} and T_{ML}/T_{c2} equal to T_{mo}/T_{mL}. Here T_{mR_1} and T_{ML} are the temperatures for which plastic flow impends during loading and unloading, respectively, and T_{mo} is the maximum temperature attainable at the insulated boundary.

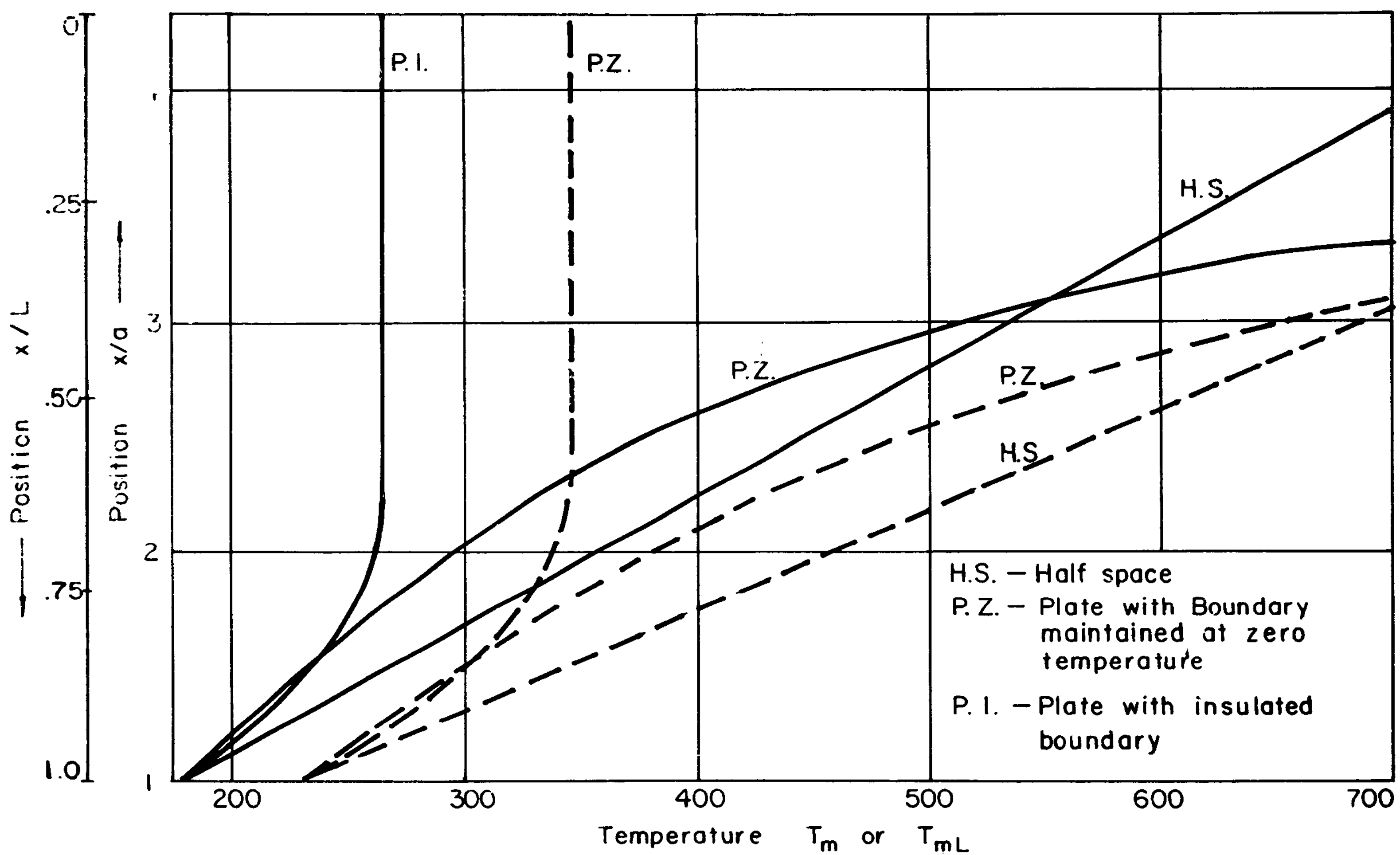

FIG. 7-2 THE DIMENSION OF THE TOTAL PLASTICALLY DEFORMED REGION WITH TEMPERATURE T_m or T_{mL}

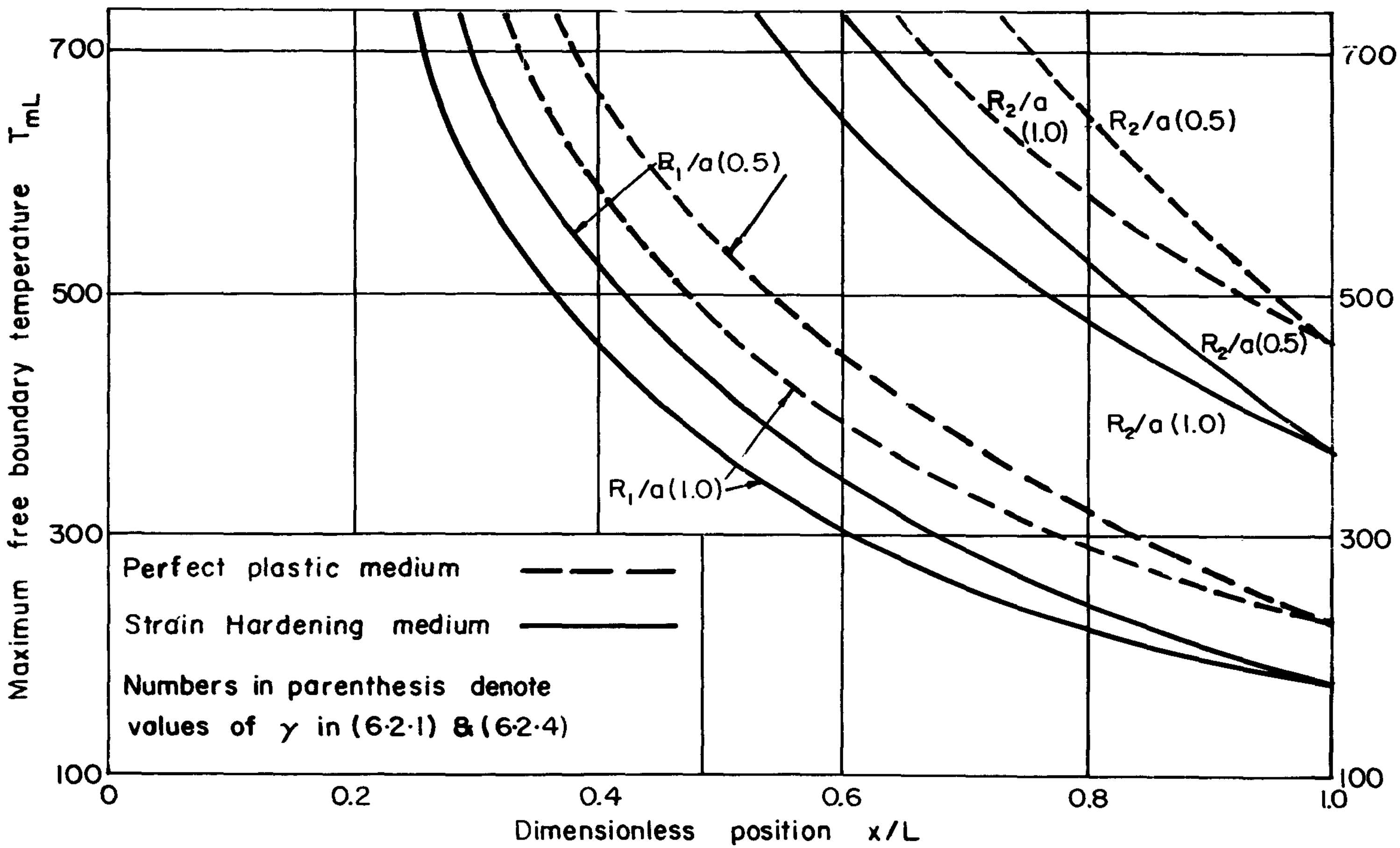

FIG. 7-3 VARIATION OF THE DIMENSIONS OF STEADY STATE PLASTIC REGION AND TOTAL PLASTICALLY DEFORMED REGION WITH MAXIMUM TEMPERATURE OF FREE BOUNDARY

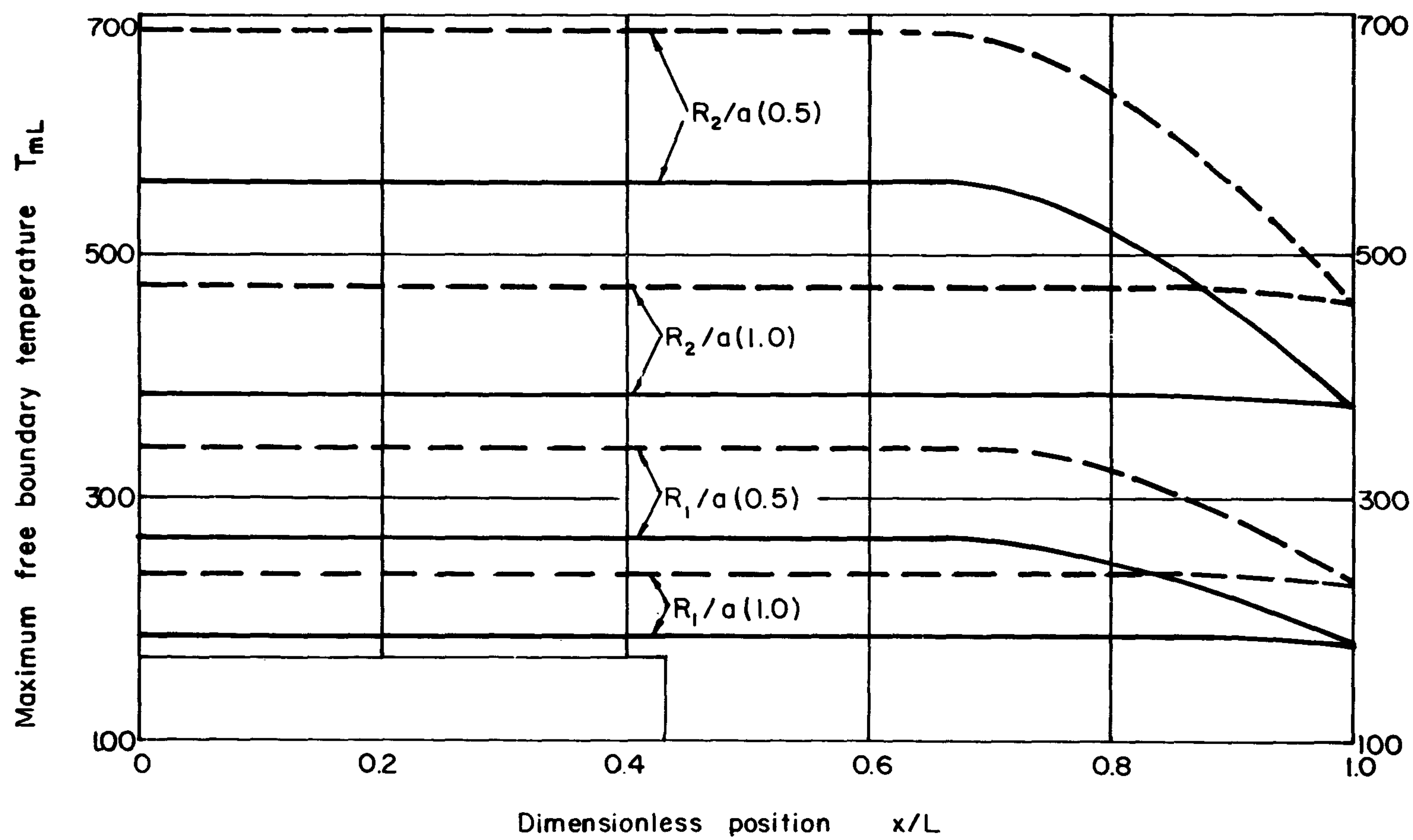

FIG. 7-4 VARIATION OF THE DIMENSIONS OF THE STEADY STATE PLASTIC REGION AND THE TOTAL PLASTICALLY DEFORMED REGION WITH MAXIMUM FREE BOUNDARY TEMPERATURE.

For the numerical values

$$\gamma = 0.50 \quad , \quad T_{mo}/T_{mL} = 0.6674 \quad ,$$

we obtain

$$T_{c1} = 266.7066^{\circ}F \quad , \quad T_{c2} = 561.9718^{\circ}F \quad , \tag{7.2.3}$$

for the strain hardening medium and

$$T_{c1} = 344.6209^{\circ}F \quad , \quad T_{c2} = 689.2418^{\circ}F \quad , \tag{7.2.4}$$

for the perfectly plastic medium.

The next calculation concerns the stress and deformation of the half-space; the transient positions of the interfaces ρ_1, ρ_2 and ρ_3 were calculated from (4.4.5), (4.5.1), (4.6.26) and (5.2.27). The results are given in Fig. 7-5, the maximum temperature T_m being used to label the curves.

Next, the equations (4.4.2), (4.5.2), (4.6.2a) were used to determine the instants t_1, t_{mR_1} and t_{pa} for the values 300°F, 400°F, 500°F, 600°F and 700°F of the maximum boundary temperature T_m. The dimensionless results of this computation together with R_1 and R_{11} for the above mentioned temperatures are presented in Table 7-1.

It was shown in chapter 4 that, if the maximum boundary temperature T_m is greater than the value T_M given by (4.6.3), and if the instant t_{mR_1} is greater than the instant t_{pa}, then four regions exist simultaneously between the instants t_{pa} and t_{mR_1} in the half-space. This may

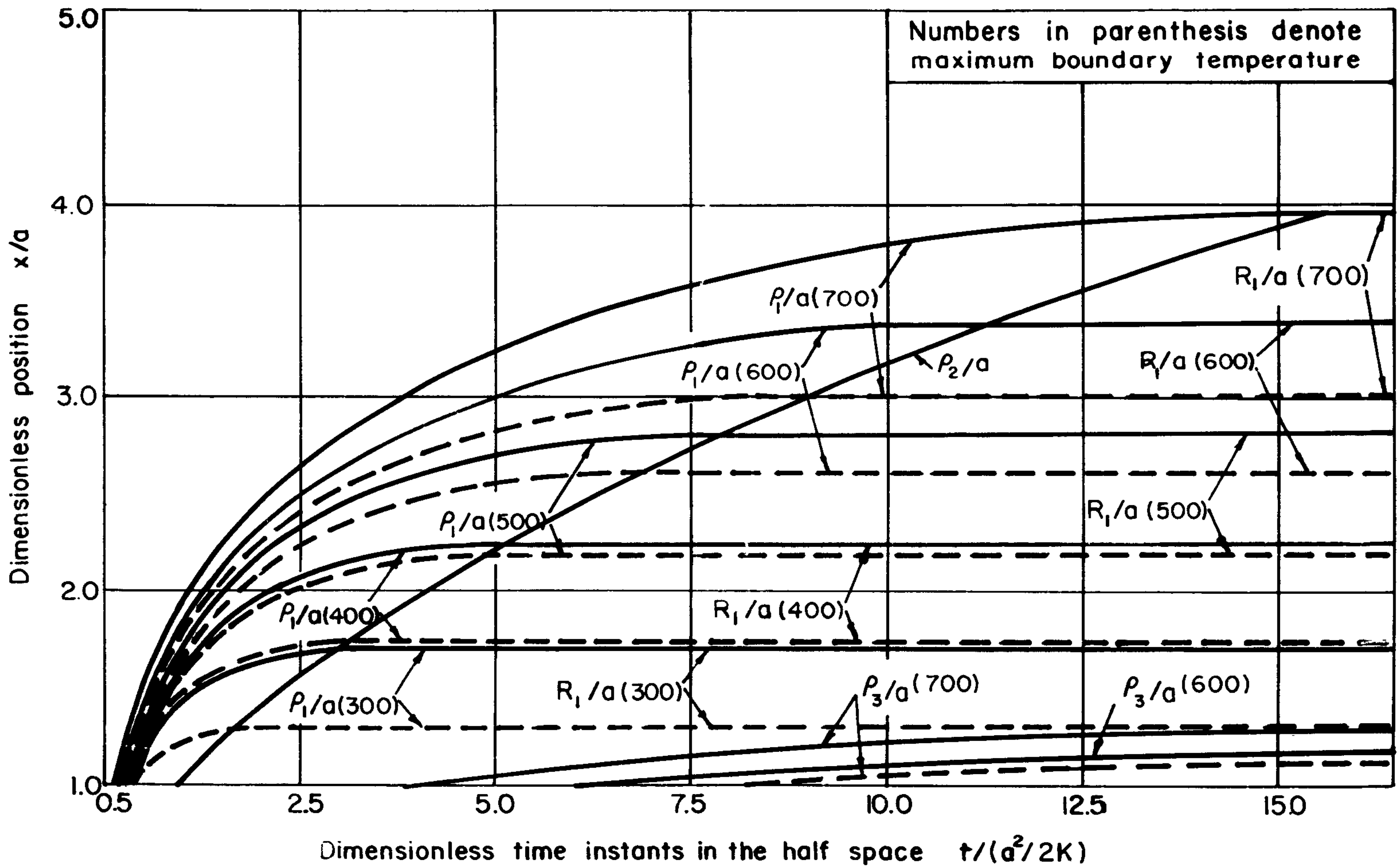

FIG. 7-5 - POSITIONS OF DIFFERENT INTERFACES AT DIFFERENT INSTANTS

Table 7-1 Characteristic Time Instants and Steady State Interface Positions

	Strain Hardening Medium with Temperature Dependent Yield Stress					Perfectly Plastic Medium with Constant Yield Stress				
T_m	n_1	n_{mR_1}	R_1/a	n_{pa}	R_{11}/a	n_1	n_{mR_1}	R_1/a	n_{pa}	R_2/a
300	0.3130	2.8495	1.6850			0.4120	1.7000	1.3040	---	---
400	0.2485	5.0700	2.2442	255.8100	1.0668	0.3050	3.0101	1.7380	---	---
500	0.2175	7.9000	2.8200	15.0020	1.3335	0.2525	4.7300	2.1770	152.3570	1.0880
600	0.1985	11.4000	3.3800	6.1500	1.6002	0.2270	6.8000	2.6050	17.9250	1.3025
700	0.1845	15.4500	3.9400	3.9115	1.8669	0.2085	9.2900	3.0420	7.4350	1.5210

R_1/a and R_{11}/a denote the dimensionless positions of the boundary of the total plastically deformed and steady state plastic regions, respectively. n_1 and n_{mR_1} are defined as the dimensionless time instants at which plastic flow during loading impends at $x = a$ and $x = R_1$; n_{pa} is defined as the dimensionless time instant at which incipient plastic flow during unloading occurs at $x = a$.

be verified in Fig. 7-5 for values of T_m equal to 600 F and 700 F for the strain hardening material, and for T_m equal to 700 F for the perfectly plastic material.

We observe from Fig. 7-5, or from (4.4.7), that the same maximum boundary temperature T_m will affect plastically a larger region if the yield stress is temperature dependent. The same observation is valid, of course, also for plates.

Making use of Fig. 7-5, and letting $T_m = 400$ F, stresses and deformations were calculated from (4.3.9), (4.3.8), (4.3.15) for the elastic region, from (4.4.26), (4.4.27), (4.4.28), (5.3.1), (5.3.2), (5.3.3) for the plastic regions, from (4.5.5), (4.5.6), (4.5.7), (5.3.4), (5.3.5), (5.3.6) and (5.3.7) for the elastic region of unloading in both the strain hardening and perfectly plastic media. The dimensionless results of the calculation are presented in Figs. 7-6 to 7-11. As time goes on, the transient stresses and deformations tend to their steady state values which were computed from (4.5.12), (4.5.13), (4.5.14), (5.3.8), (5.3.9), (5.3.10) for the steady state elastic region, and from (4.6.11), (4.6.12), (4.6.13), (5.3.8), (5.3.9), (5.3.10) for the steady state plastic region. The dimensionless results of the computation for different values of the maximum boundary temperature T_m for the residual stress, strain and displacement are shown in Figs. 7-12, 7-13, and 7-14, respectively.

Fig. 7-6 shows that the largest compressive stress induced in the plastic region is greater in the perfectly plastic medium than in the strain hardening medium having a temperature dependent yield stress. The same is not true, however, for residual tensile stress. This is

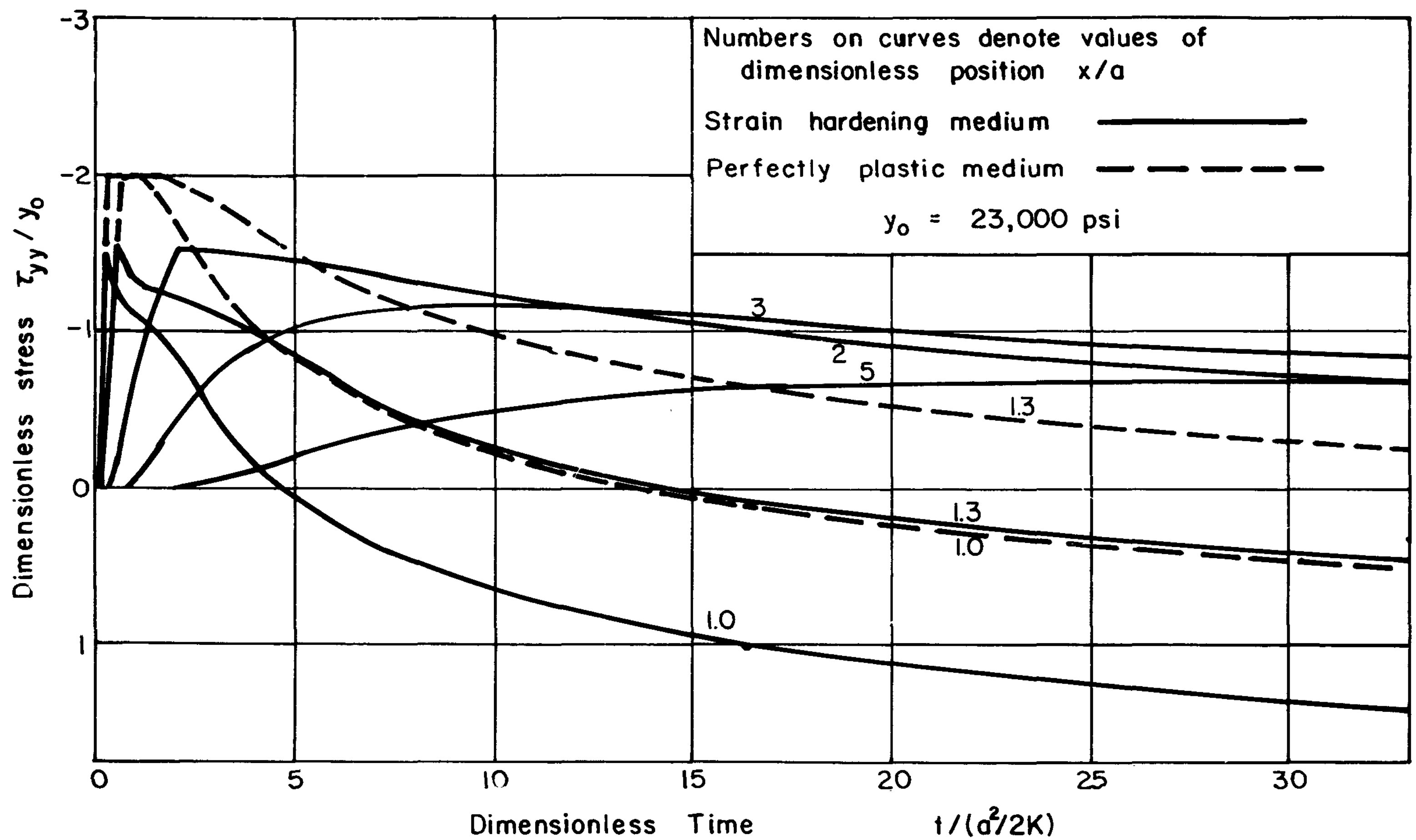

FIG. 7-6 VARIATION OF STRESS WITH TIME AT DIFFERENT POSITIONS IN THE HALF SPACE

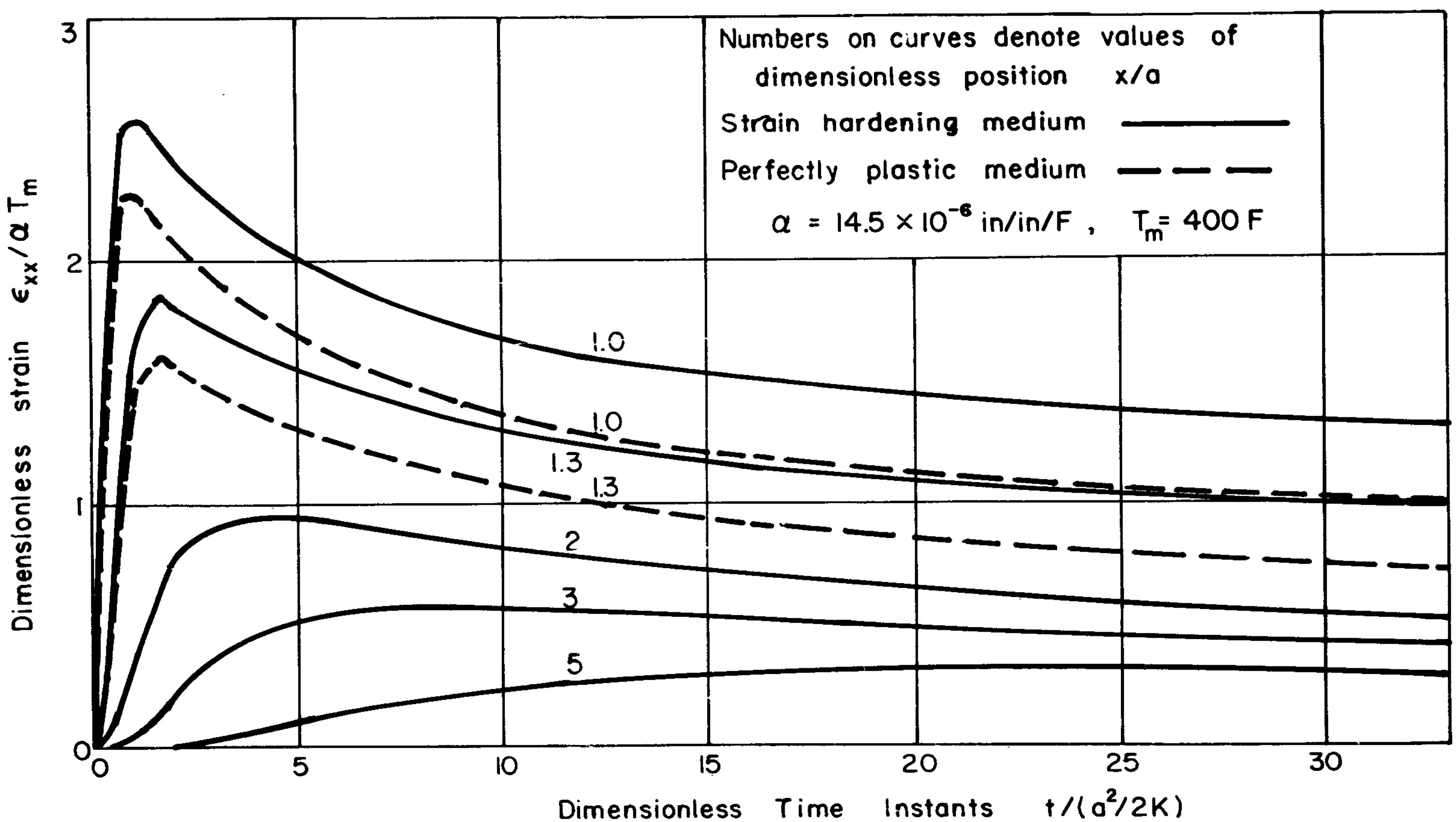

FIG 7-7 VARIATION OF STRAIN WITH TIME AT DIFFERENT POSITIONS IN THE HALF SPACE.

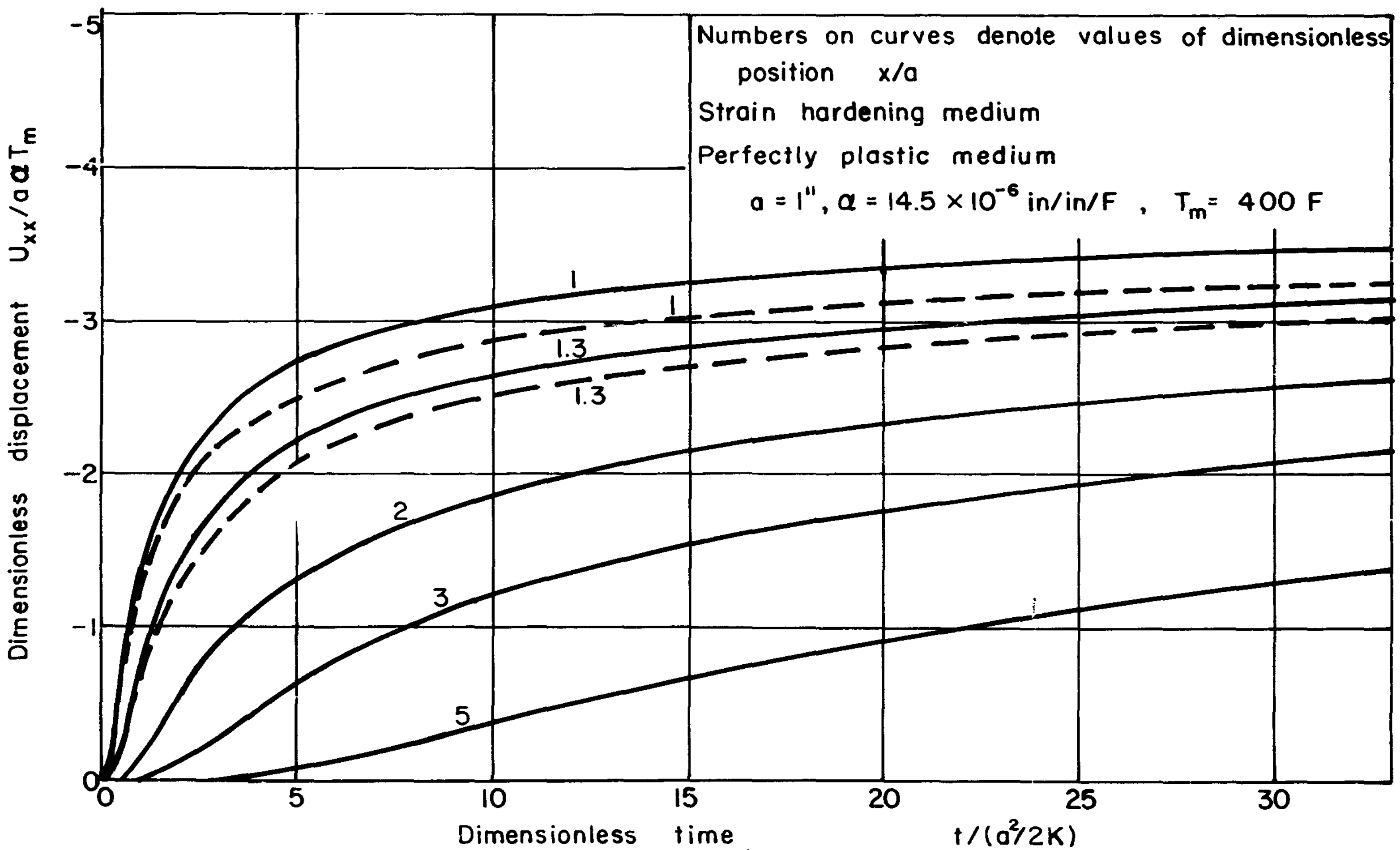

FIG 7-8 VARIATION OF DISPLACEMENT WITH TIME AT DIFFERENT POSITIONS IN THE HALF SPACE

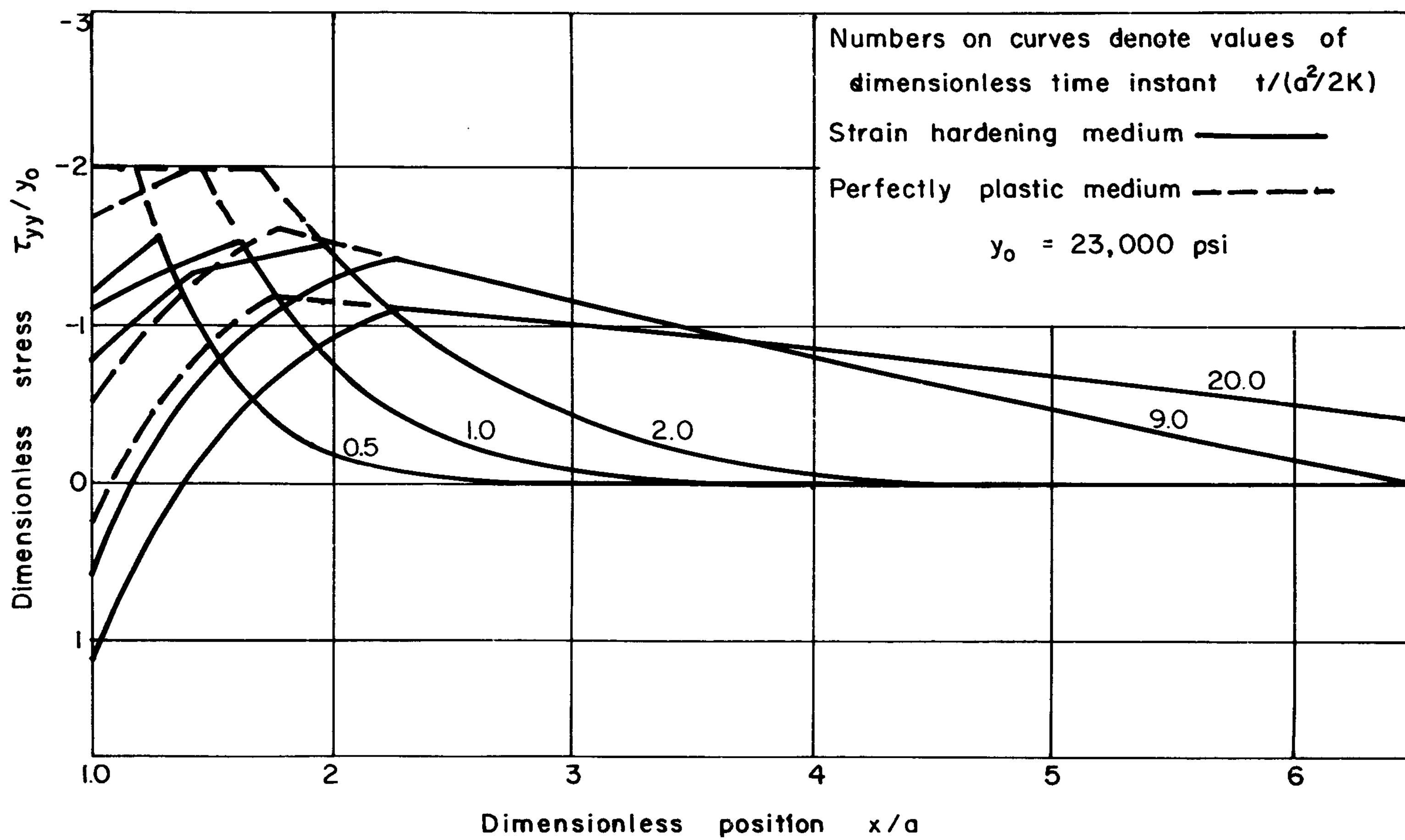

FIG 7-9 STRESS DISTRIBUTION AT DIFFERENT TIME INSTANTS IN THE HALF SPACE

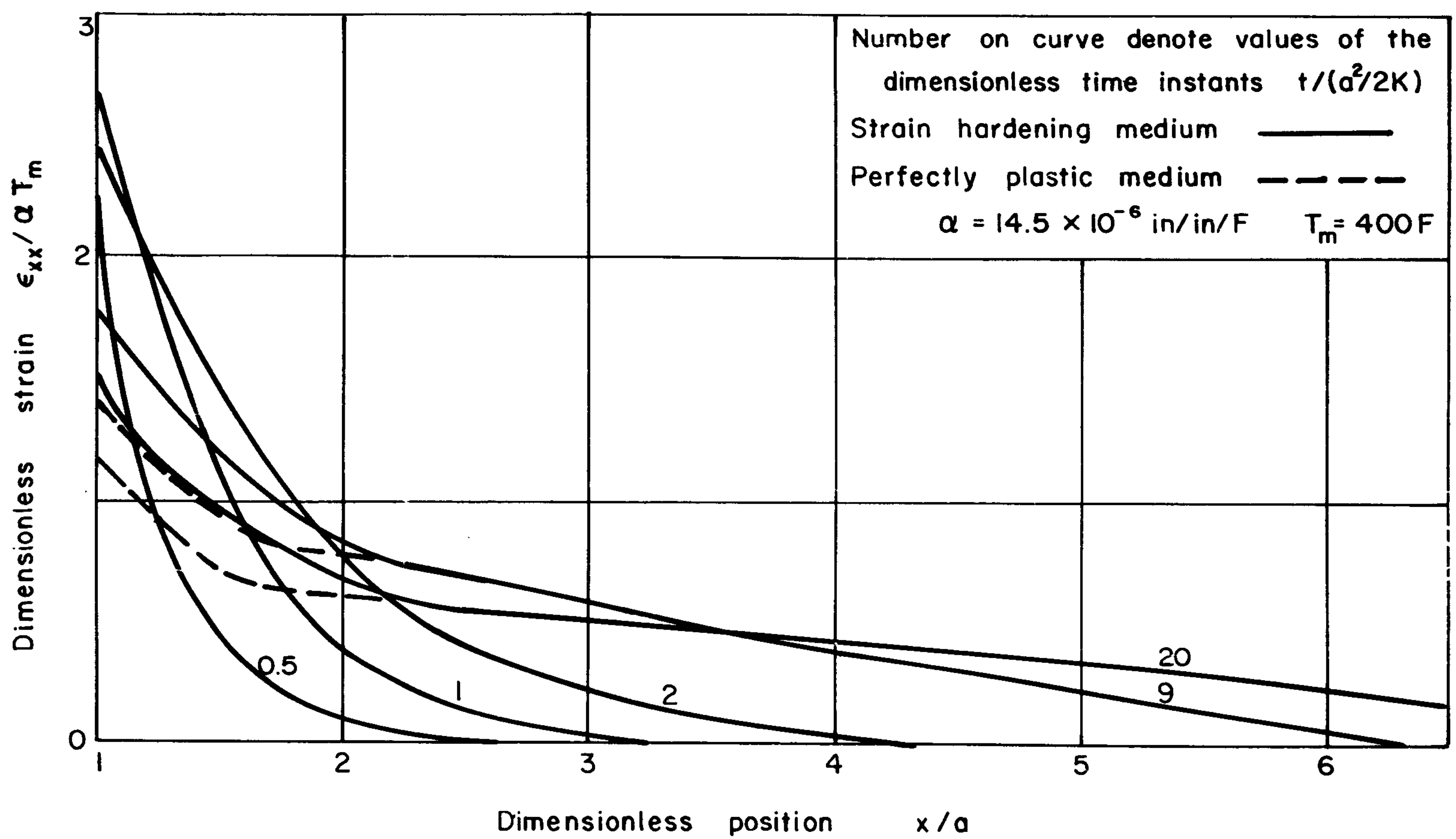

FIG 7-10 STRAIN DISTRIBUTION AT DIFFERENT INSTANTS IN THE HALF SPACE

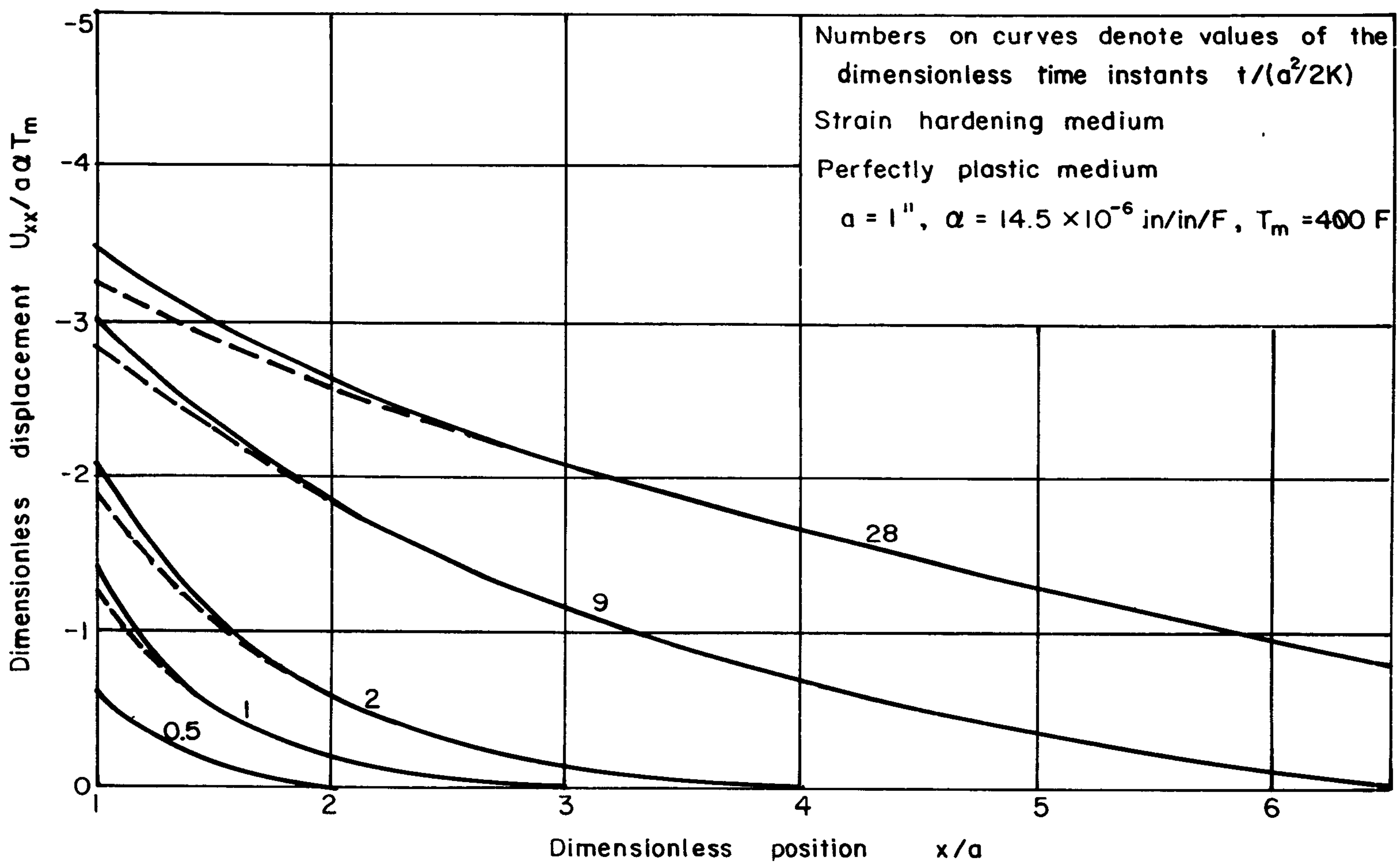

FIG 7-11 DISPLACEMENTS AT DIFFERENT TIME INSTANTS IN THE HALF SPACE

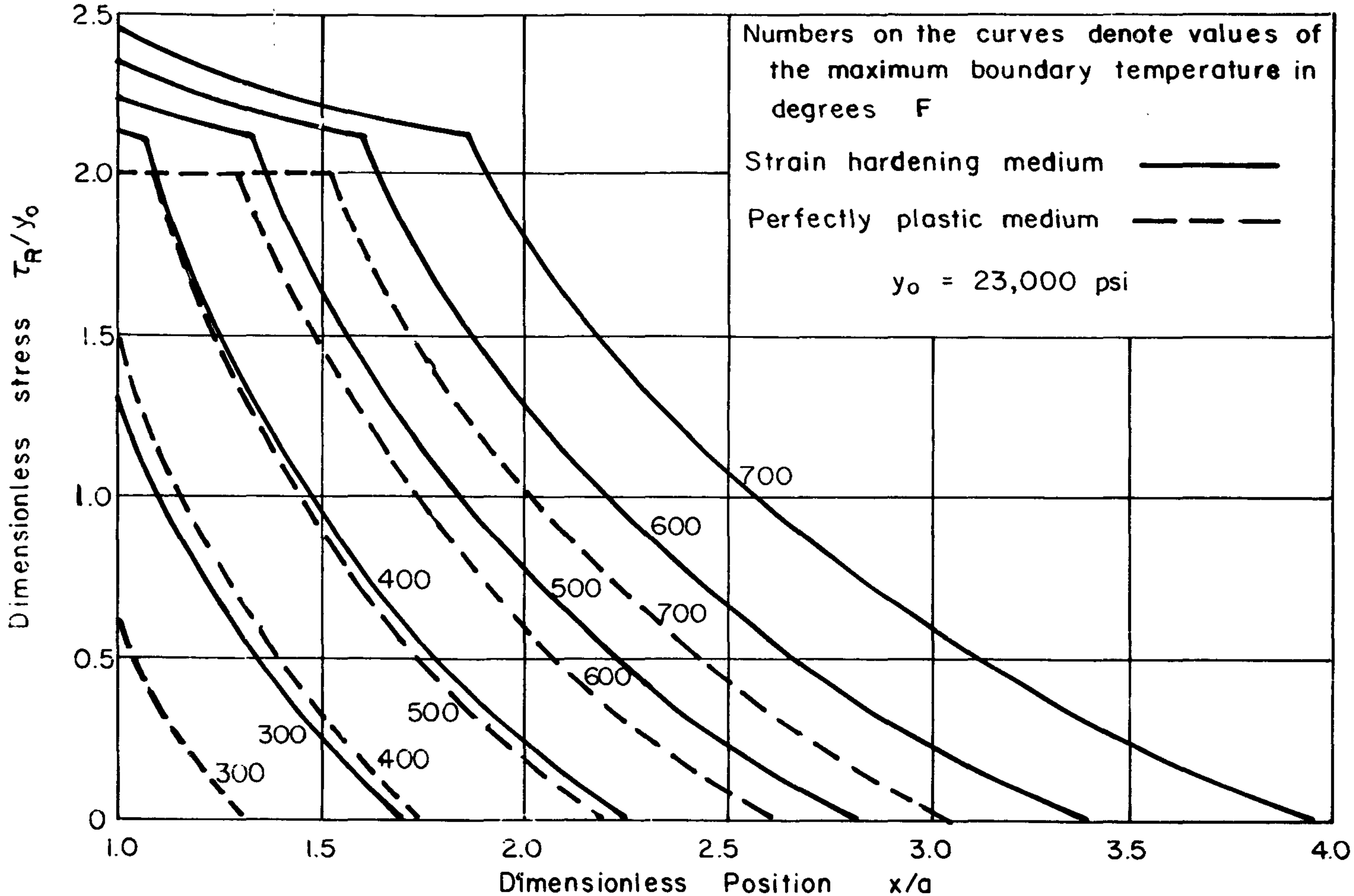

FIG. 7-12 RESIDUAL STRESS DISTRIBUTION IN THE HALF SPACE

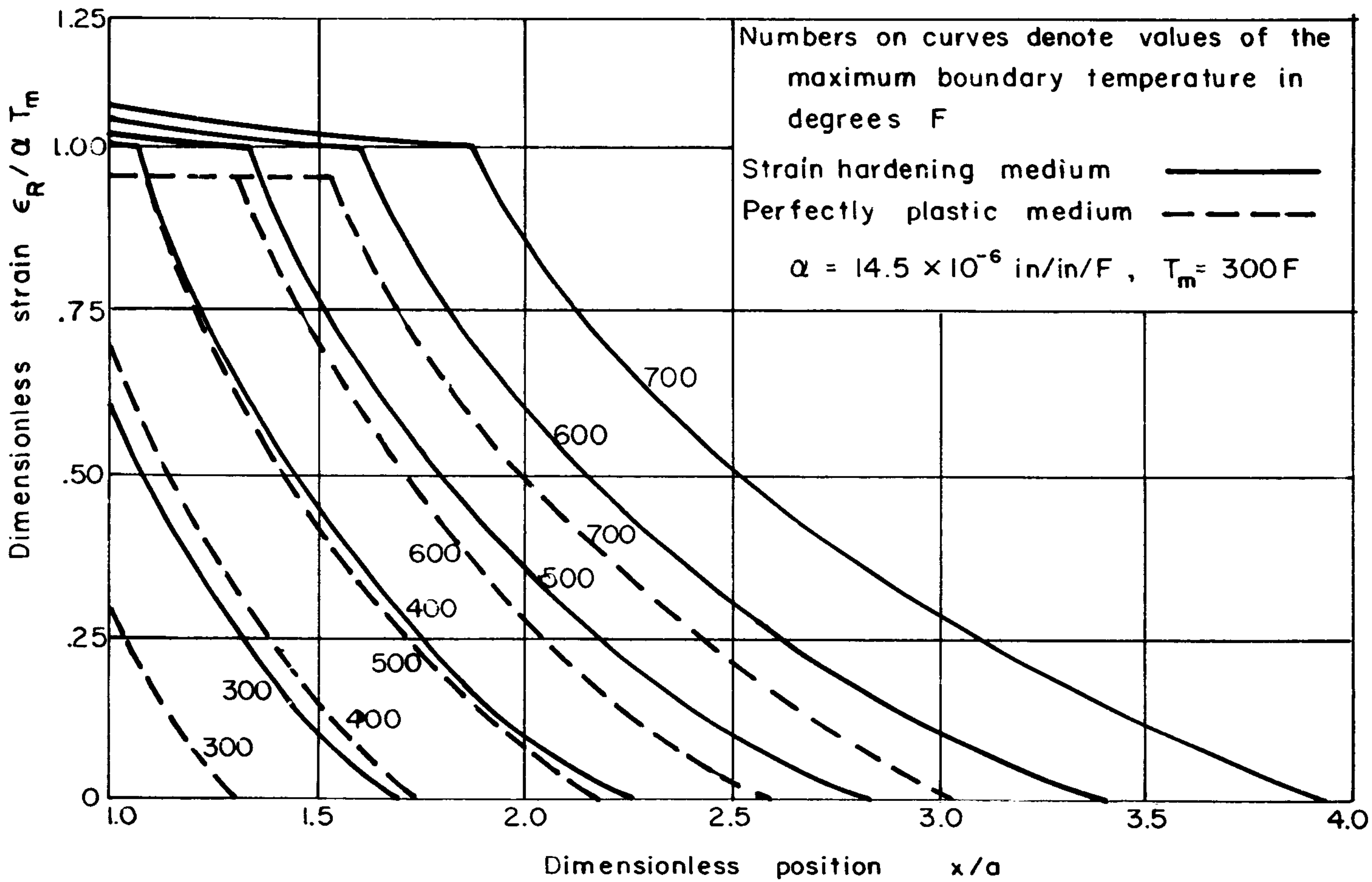

FIG 7-13 RESIDUAL STRAIN DISTRIBUTION IN THE HALF SPACE

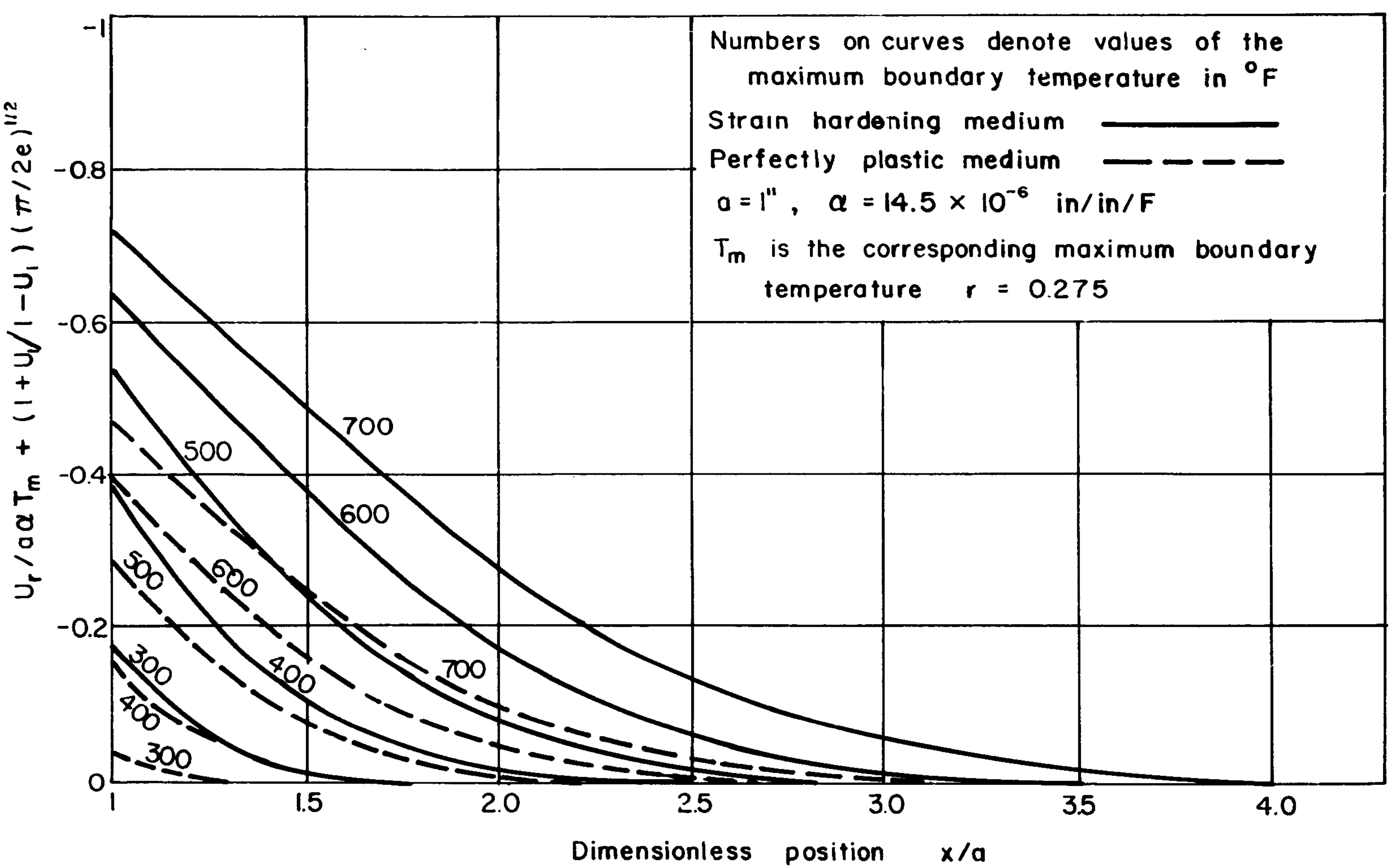

FIG 7-14 PERMANENT DEFORMATION IN THE HALF SPACE

due to the fact that the transient stresses in both the strain hardening and perfectly plastic media, after falling back from their maximum value in compression to zero, begin to increase in the opposite direction; the transient tensile stress in the strain hardening medium is at any instant higher than that in the perfectly plastic medium (Fig. 7-6) and eventually assumes a steady state value also higher than the one assumed by the perfectly plastic medium. The transient and residual strains and displacements are, however, greater in the strain hardening medium, as may be seen from Figs. 7-7, 7-8, 7-10, 7-11, 7-13, and 7-14. Another point worth noting is that in both the transient and steady state solutions (Figs. 7-9 to 7-14) the stresses, strains and displacements are always greatest at the boundary. We further observe from Figs. 7-12 and 7-13 that the residual elastic stress and strain increase much more rapidly with the maximum boundary temperature T_m than their plastic counterparts in the total plastically deformed region; the position R_{11} of the steady state elastic plastic interfaces can be read off from these figures by locating the planes at which there is an abrupt change in the slope of the curves. For $T_m = 400$ F, the maximum transient and residual stresses, strains and displacement were found to be as follows:

$$\begin{aligned} &\tau_y/y_o = -1.5488 , && \text{at } n = n_1 = 9.2485 , \\ &\epsilon_{xx}/\alpha T_m = 2.5840 , && \text{at } n = 1.00 , \qquad (7.2.5) \\ &\tau_R/y_o = 2.3767 , && \epsilon_R/\alpha T_m = 0.8295 , \; U_R/a\alpha T_m = -4.0147 \end{aligned}$$

for the strain hardening medium and

$$\tau_y/y_o = -2.000 \quad , \qquad \text{at} \quad n = n_1 = 0.3050 \quad ,$$

$$\epsilon_R/\alpha\, T_m = 2.2860 \quad , \qquad \text{at} \quad n = 1.0000 \tag{7.2.6}$$

$$\tau_R/y_o = 1.4800 \quad , \quad \epsilon_R/\alpha\, T_m = 0.5174 \quad , \quad U_R/a\alpha\, T_m = -3.7950$$

for the perfectly plastic medium.

Recalling the discussion presented in section 7-1, we conclude that the values given by (7.2.3), (7.2.4) for the maximum transient and residual stresses and strains also represent the corresponding quantities induced in a plate by a heat pulse of the same maximum amplitude, i.e., for $T_{mL} = 400$ F.

The transient displacements are shown in Fig. 7-8; the fact that even in the elastic region the residual displacement does not vanish is due to the regularity condition† used. Therefore, in presenting the permanent deformations in the half-space, the quantity

$$[U_R/a\alpha\, T_m + (1+\nu_1/1-\nu_1)\ \sqrt{\tfrac{\pi}{2}\, e}]$$

was plotted against position x/a (Fig. 7-14). The resulting curves are found to decrease from a maximum value at the boundary of the half-space to zero at the boundary of the total plastically deformed region.

7-3 Further Discussion

In the preceding sections of this chapter we determined the dimensions of the total plastically deformed and steady state plastic regions and also obtained expressions for the residual stresses and

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† cf. see equation (4.3.6)

strains caused by a uniformly applied heat pulse to the boundary of a half-space and a plate. We are now confronted with the problem of predicting the response of the medium to an additional heat pulse applied after conditions produced by the first pulse have reached a steady state. In particular, we would like to find the amplitudes of the second heat pulse for which plastic flow will recur.

Let us consider the application of a second heat pulse in some detail, assuming that steady state condition has been reached for the first pulse. The relation between maximum shearing stress q and temperature T is given by

$$\dot{q} = \frac{\alpha E_1}{2(1-\nu_1)} \dot{T} \quad , \tag{7.4.1}$$

where the maximum shearing stress q will increase from its residual value q_R as the temperature T increases from Zero. Thus integration of (7.4.1) yields the relation

$$q = \frac{\alpha E_1}{2(1-\nu_1)} T + q_R \quad . \tag{7.4.2}$$

Recalling,

$$q = -\frac{\tau_y}{2} \quad , \qquad q_R = -\frac{\tau_R}{2} \quad , \qquad \text{(4.3.12) R}$$

we may write (7.4.2) as

$$q = \frac{\alpha E_1}{2(1-\nu_1)} T - \frac{\tau_R}{2} \tag{7.4.3}$$

where τ_R is the residual stress for which the distribution in the half-space is shown in Fig. 7-12. Plastic flow due to biaxial compression will resume wherever the maximum shearing stress q given by (7.4.3) becomes equal to the current yield stress y_c in shear, where

$$y_c = y_o - BT + \frac{1}{1-s} 2\mu_p p'' \tag{7.4.4}$$

The relation (7.4.4) is actually the relation (f) in section 4-42.

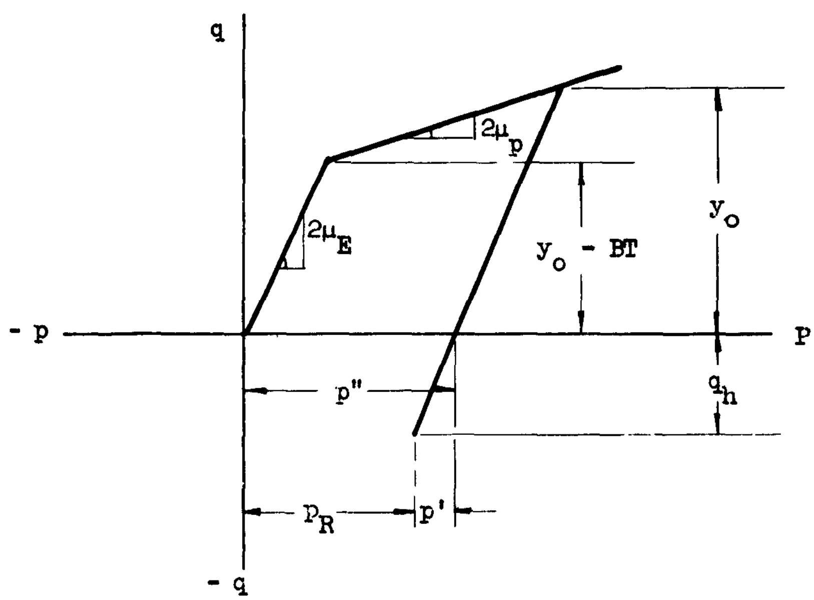

Fig. 7-15 Shearing Stress, Strain and Current Yield Stress

From Fig. 7-15 we observe that

$$p'' = p_R + p' \quad ,$$

p_R being the principal residual shearing strain,

$$p' = - \frac{q_R}{2\mu_E} \quad .$$

If we recall that

$$p_R = \frac{\epsilon_R}{2} \quad , \quad q_R = - \frac{\tau_R}{2} \quad , \tag{4.3.12 R}$$

then the expression for the current yield stress becomes

$$y_c = y_o - BT + \frac{1}{1-s} \left(2\mu_p \frac{\epsilon_R}{2} + s \frac{\tau_R}{2} \right) \quad . \tag{7.4.5}$$

Replacing q in (7.4.3) by y_c given by (7.4.5), we find the temperature T_{1x}' above which plastic flow will again occur as the result of the second heat pulse:

$$T_{1x}' = \frac{y_o}{B + \alpha E_1/2(1-{}_1)} \left[1 + 1/2 \frac{\tau_R}{y_o} \left(1 + \frac{s}{1-s}\right) + \frac{\mu_p}{1-s} \frac{\epsilon_R}{y_o}\right] \quad . \tag{7.4.6}$$

As may be readily seen from (7.4.6), the temperature T'_{1x} is not constant but varies with the residual stress and strain in the total plastically deformed region. In the region that has never been plastically deformed, the residual stress τ_R and strain ϵ_R are zero, and, in this case, (7.4.6) reduces to the form

$$T_{1x}' = \frac{y_o}{B + \alpha E_1/2(1-\nu_1)} \tag{7.4.7}$$

identical with (4.4.6). Therefore, as long as the medium remains in the virgin state, the temperature above which plastic flow is incipient remains equal to T_{1x}, regardless of the number of pulses to which the medium may have been subjected.

With the values of the residual stress and strain given in Figs. 7-12 and 7-13 for the half-space, and the values of the maximum residual stress and strain obtainable from Figs. 7-1 and 7-13 for the plate, the temperature T'_{1x} for the strain hardening medium, and corresponding to different values of T_m or T_{mL}, was calculated, the results being summarized in Table 7-2.

Table 7-2 Temperatures Corresponding to Recurrence of Plastic Flow at the Heated Boundary of Half-Space or Plate

$T_m = T_{mL}$	$300^\circ F$	$400^\circ F$	$500^\circ F$	$600^\circ F$	$700^\circ F$
$T_{1x} = T_{mR_1}$ ($^\circ F$)	178	178	178	178	178
T_{1x}' ($^\circ F$)	309.9692	393.7360	405.5055	414.6332	425.0640

The values of T'_{1x} for other positions in the medium can also be determined readily by direct substitution of the corresponding values of τ_R and ϵ_R into (7.4.6). Comparing (4.4.6) with (7.4.6), we observe that as a result of the first heat pulse, the temperature corresponding to the incipient yielding in the plastically deformed region has been raised by a factor of

$$[1 + 1/2 \frac{\tau_R}{y_o} (1 + \frac{s}{1-s}) + \frac{\mu_p}{1-s} \frac{\epsilon_R}{y_o}] \quad .$$

We further note from the results presented in Table 7-2 that if the maximum amplitude T_m or T_{mL} of the first heat pulse remains within certain limit, no further plastic flow will be induced by the second heat pulse having the same amplitude as the first one. The case where $T_m = 300$ F may serve to illustrate this point (Table 7-2). On the other hand, if the maximum amplitude of the first pulse exceeds this limit, and the maximum amplitude of the second pulse is equal to that of the first pulse, further plastic flow will be induced, and plastic strains will tend to accumulate. This contention is borne out by an inspection of the cases where T_m or T_{mL} is equal to $400^\circ F$, $500^\circ F$, $600^\circ F$ and $700^\circ F$ (Table 7-2).

For a more detailed determination of the plastic flow during a second heat pulse a complete analysis, comparable to that presented for the first heat pulse, is needed. Although we have not pursued this analysis, the procedures outlined in this work are applicable and sufficient for the study of any subsequent heat pulses.

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APPENDIX A

Maximum Temperature Distribution Curves Used for the Determination of the Positions R_1 and R_{11} of the Total Plastically Deformed and Steady State Plastic Region in Plates. (Figs. 7-3, 7-4).

Fig. A-1 For the Plate With the Boundary at $x = L$ Subjected to a Heat Pulse and With the Other Boundary at $x = o$ Maintained at Zero Temperature.

Fig. A-2 For the Plate With the Boundary at $x = L$ Subjected to a Heat Pulse and With the Other Boundary at $x = o$ Insulated.

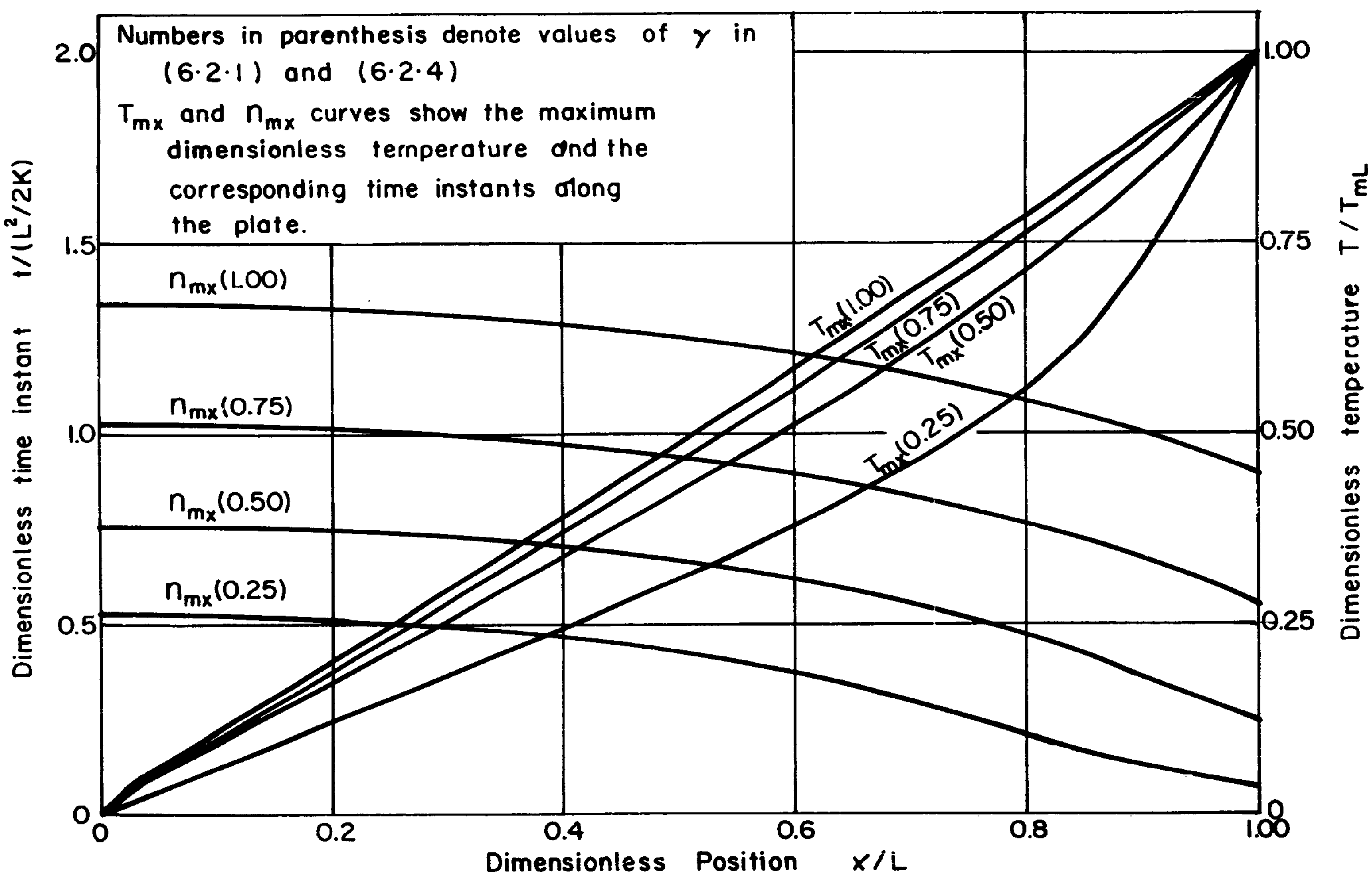

FIG. A-I VARIATION OF MAXIMUM TEMPERATURE AND CORRESPONDING INSTANTS ALONG PLATE

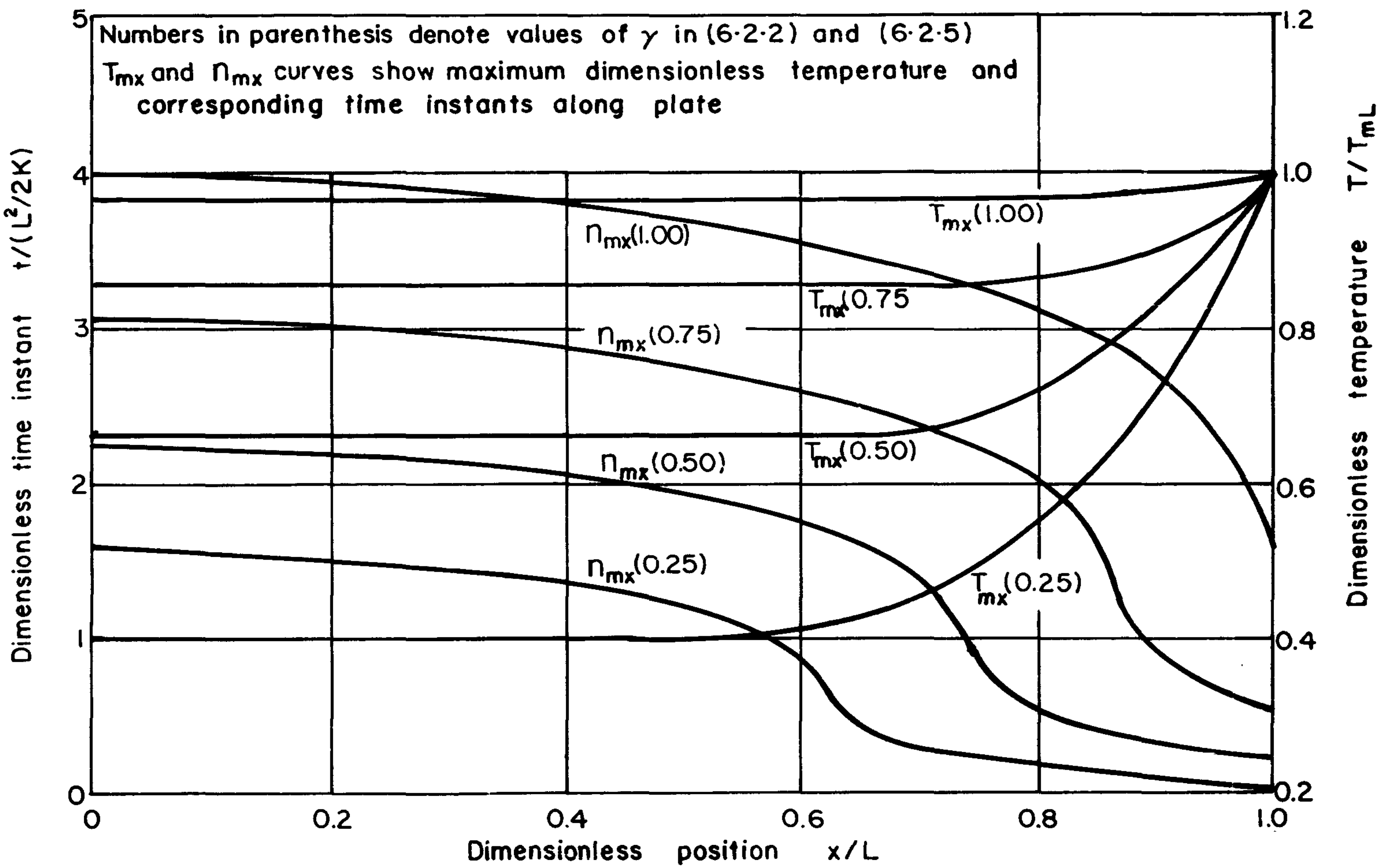

FIG. A-2 VARIATION OF MAXIMUM TEMPERATURE AND CORRESPONDING INSTANTS ALONG PLATE

APPENDIX B

<u>Derivation of Plastic Stress Strain Relations s_{xx} and s_{yy} Given by (a) and (b) in (4.4.14)</u>

The relation (4.4.13) in principal stress space may be written as

$$s_i = \sum_{\gamma=1}^{2} s_i^{(\gamma)} \quad ,$$

$$s_i^{(\gamma)} = \frac{\partial f^{(\gamma)}/\partial\tau_i}{\partial H/\partial\xi \quad (\partial f^{(\gamma)}/\partial\tau k \quad \partial f^{(\gamma)}/\partial\tau k)^{1/2}} \qquad (4.4.13)\text{ R}$$

$$(\partial F^{(\gamma)}/\partial v_m \quad \dot{v}_m + \partial F^{(\gamma)}/\partial T \; \dot{T})$$

the yield functions are given by

$$F^{(1)} = \frac{v_x - v_y}{2} - (y_2 - BT + \frac{1}{1-s} \; 2\mu_p \frac{\xi}{\sqrt{6}}) \quad , \qquad (4.4.8)\text{ R}$$

$$F^{(2)} = \frac{v_x - v_z}{2} - (y_o - BT + \frac{1}{1-s} 2\mu_p \frac{\xi}{\sqrt{6}}) \quad , \qquad (4.4.9)\text{ R}$$

$$f^{(1)} = \frac{v_x - v_y}{2} = - \frac{\tau_{yy}}{2} \quad ,$$

$$f^{(2)} = \frac{v_x - v_z}{2} = - \frac{\tau_{zz}}{2} \quad , \qquad (4.4.10)\text{ R}$$

$$H = (y_o - BT + \frac{1}{1-s} \; 2\mu_p \frac{\xi}{\sqrt{6}})$$

Recalling that

$$\tau_{xx} = 0 \quad , \quad \tau_{yy} = \tau_{zz}$$

and

$$v_{ij} = \tau_{ij} - 1/3\, \delta_{ij}\, \tau_{kk}$$

we have

$$v_x = -\,2/3\, \tau_{yy} = -\,2/3\, \tau_{zz} \quad ,$$

$$v_y = 1/3\, \tau_{yy} \quad , \quad v_z = 1/3\, \tau_{yy} \quad . \tag{a}$$

By (4.4.8), (4.4.9), (4.4.10) and (a), we obtain

$$\partial f^{(1)}/\partial\tau_{xx} = 1/2 \quad , \quad \partial f^{(1)}/\partial\tau_{yy} = -\,1/2 \quad ,$$

$$\partial F^{(1)}/\partial v_x = 1/2 \quad , \quad \partial F^{(1)}/\partial v_y = -\,1/2 \quad , \quad \partial F^{(1)}/\partial v_z = 0 \quad , \tag{b}$$

and

$$\partial f^{(2)}/\partial\tau_{xx} = 1/2 \quad , \quad \partial f^{(2)}/\partial\tau_{zz} = -\,1/2 \quad ,$$

$$\partial F^{(2)}/\partial v_x = 1/2 \quad , \quad \partial F^{(2)}/\partial v_y = 0, \quad \partial F^{(2)}/\partial v_z = -\,1/2 \quad . \tag{c}$$

By (4.4.8), (4.4.9) and (4.4.10) it was found that

$$\partial H/\partial\xi = (1/1\text{-}s)\sqrt{2/3}\;\mu_p \quad ,$$

$$\partial F^{(1)}/\partial T = \partial F^{(2)}/\partial T = B \quad . \tag{d}$$

By (a), (b) and (d), the following results are obtained:

$$(\partial f^{(1)}/\partial \tau_k \;\; \partial f^{(1)}/\partial \tau_k)^{1/2}$$

$$= (\partial f^{(1)}/\partial \tau_{xx} \; \partial f^{(1)}/\partial \tau_{xx} + \partial f^{(1)}/\partial \tau_{yy} \; \partial f^{(1)}/\partial \tau_{yy} + \partial f^{(1)}/\partial \tau_{zz} \; \partial f^{(1)}/\partial \tau_{zz})^{1/2}$$

$$= [1/2 \; 1/2 + (-1/2)(-1/2) + 0 \; 0]^{1/2} = (1/2)^{1/2}, \qquad \text{(e)}$$

$$(\partial F^{(1)}/\partial v_k \; \dot{v}_k + \partial F^{(1)}/\partial T \; \dot{T})$$

$$= (\partial F^{(1)}/\partial v_x \; \dot{v}_x + \partial F^{(1)}/\partial v_y \; \dot{v}_y + \partial F^{(1)}/\partial v_z \; \dot{v}_z + \partial F^{(1)}/\partial T \; \dot{T})$$

$$= [1/2 \; (-2/3 \;\; \dot{\tau}_{yy}) + (-1/2)(1/3 \; \dot{\tau}_{yy}) + 0 \; (1/3 \; \dot{\tau}_{yy}) + B\dot{T}]$$

$$= -1/2 \; \dot{\tau}_{yy} + B\dot{T} \; . \qquad \text{(f)}$$

Similarly, it was found that

$$(\partial f^{(2)}/\partial \tau_k \quad \partial f^{(2)}/\partial \tau_k)^{1/2} = (1/2)^{1/2} \qquad \text{(g)}$$

and

$$(\partial F^{(2)}/\partial v_k \; \dot{v}_k + \partial F^{(2)}/\partial T \; \dot{T})$$

$$= -1/2 \; \dot{\tau}_{yy} + B\dot{T} \qquad \text{(h)}$$

Substituting (b), (c), (d), (e), (f), (g), and (h) in (4.4.13), we finally obtain

$$s_{xx} = s_{xx}^{(1)} + s_{xx}^{(2)}$$

$$= \frac{1}{\frac{1}{1-s}\,\mu_p\,(1/3)^{1/2}} \left(- 1/2\ \dot{\tau}_{yy} + B\dot{T} \right) \qquad \text{(a) in (4.4.14)}$$

$$s_{yy} = s_{yy}^{(1)} = s_{zz} = s_{zz}^{(2)}$$

$$= - 1/2\ \frac{1}{\frac{1}{1-s}\,\mu_p\,(1/3)^{1/2}} \left(- 1/2\ \dot{\tau}_{yy} + B\dot{T} \right)$$

(b) in (4.4.14)

APPENDIX C

Reduction of the Strain Hardening Solution to Perfectly Plastic Solution

It will be shown in what follows that (5.2.6) is reducible from (4.4.16).

We recall that

$$\dot{\epsilon}_x = [\frac{4(1-2\nu_1)}{E_1} B + 3\alpha] \dot{T} \quad , \qquad (5.2.6)R$$

and

$$\dot{\epsilon}_x = N \dot{T} \quad , \qquad (4.4.16)R$$

where

$$N = [\alpha + \frac{B (1-s) \sqrt{3}}{\mu_p}] - M [\frac{(1-s) \sqrt{3}}{2\mu_p} - \frac{2\nu_1}{E}] \quad ,$$

$$M = \frac{2B E_1 (1-s) - 2 E_1 \mu_p \sqrt{4/3}}{E_1 (1-s) + 2(1-\nu_1) \mu_p \sqrt{4/3}} \quad . \qquad (4.4.17)R$$

We now define

$$N_{pp} = [\frac{4(1-2\nu_1)}{E_1} B + 3\alpha]. \qquad (a)$$

If μ_p in (4.4.17) is set equal to zero, N should be reducible to N_{pp}. N may be rewritten as

$$N = \alpha - \frac{2\nu_1}{E_1} M + \sqrt{3} (1-s) \frac{2B - M}{2 \mu_p} \qquad (b)$$

Substituting M given by (4.4.17) in the quantity $(2B-M)/2\mu_p$, we eventually have

$$\frac{2B - M}{2\,\mu_p} = \sqrt{4/3}\;\frac{2B\,(1-\nu_1) + E_1\,\alpha}{E_1(1-s) + 2(1-\nu_1)\,\mu_p\sqrt{4/3}} \qquad (c)$$

If μ_p is set equal to zero, corresponding to a perfectly plastic material, the following results are obtained.

$$M = 2B \quad , \quad s = \mu_p/\mu_E = 0$$

$$\frac{2B - M}{2\,\mu_p} = \sqrt{4/3}\;\frac{2B(1-\nu_1) + E_1\,\alpha}{E_1} \qquad (d)$$

Now, if (d) is substituted back in (b), we obtain

$$N = \alpha - \frac{2\nu_1}{E_1}\,2B + \sqrt{3}\,\sqrt{4/3}\,[\frac{2B(1-\nu_1)}{E_1} + \alpha]$$

$$= \alpha - \frac{4\nu_1}{E_1}\,B + \frac{4(1-\nu_1)}{E_1}\,B + 2\,\alpha$$

$$= [\frac{4(1-2\nu_1)}{E}\quad B + 3\,\alpha] = N_{pp} \quad .$$

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